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Criteria of univalence for a certain integral operator

Abstract. In this article we consider the problem of univalence of a function introduced by Breaz and Ularu, improve some of their results and receive not only univalence conditions but also close-to-convex conditions for this function. To this aim, we used our method based on Kaplan classes.

1. Introduction. We consider the following subclasses of the class of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$:

- $\mathcal{A}$ as the class of all functions f normalized by $f(0) = f'(0) - 1 = 0$,
- $\mathcal{H}$ as the subclass of $\mathcal{A}$ of all functions f that are locally univalent, i.e., $f' \neq 0$ in $\mathbb{D}$,
- $\mathcal{S}$ as the class of all univalent functions belonging to $\mathcal{A}$,
- $\mathcal{K}$ as the class of functions in $\mathcal{S}$ that map $\mathbb{D}$ onto a convex set,
- $\mathcal{C}$ as the class of functions in $\mathcal{S}$ that are close-to-convex,
- $\mathcal{H}_d$ as the class of all analytic functions f normalized by $f(0) = 1$ and such that $f \neq 0$ in $\mathbb{D}$.

Univalence of integral operators for the functions from known classes $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{S}$ was studied by many authors (see [1, 3, 4, 6–8]). In this article we consider univalence of slightly modified integral operator introduced in [2].

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For $\alpha, \beta \geq 0$, the Kaplan class $K(\alpha, \beta)$ is a set of all functions $f \in \mathcal{H}_d$ satisfying the condition

$$(1.1) \qquad -\alpha\pi - \frac{1}{2}(\alpha - \beta)(\theta_1 - \theta_2) \leq \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})$$

for $0 < r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$ (see [9, pp. 32–33]).

Let us recall [9, p. 46] that:

- $f \in \mathcal{K}$ if and only if $f' \in K(0, 2)$,
- $f \in \mathcal{C}$ if and only if $f' \in K(1, 3)$.

First we will call the following lemmas from [5].

Lemma A. *For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ and $t \in \mathbb{R}\backslash\{0\}$ the following conditions hold:*

$$f \in K(\alpha_1, \beta_1) \text{ and } g \in K(\alpha_2, \beta_2) \Rightarrow fg \in K(\alpha_1 + \alpha_2, \beta_1 + \beta_2)\,,$$

$$f \in K(\alpha_1, \beta_1) \iff f^t \in K\left(\frac{|t| + t}{2}\alpha_1 + \frac{|t| - t}{2}\beta_1, \frac{|t| + t}{2}\beta_1 + \frac{|t| - t}{2}\alpha_1\right),$$

$$f \in K(\alpha_1, \beta_1) \Rightarrow f^0 \in K(0, 0)\,.$$

Lemma B. *For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ the following equivalences hold:*

$$\alpha_1 \leq \alpha_2 \iff K(\alpha_1, \beta_1) \subset K(\alpha_2, \beta_1)\,,$$
$$\beta_1 \leq \beta_2 \iff K(\alpha_1, \beta_1) \subset K(\alpha_1, \beta_2)\,.$$

Now we will define an integral operator which is the subject of our research.

Definition 1.1. Let $\nu \in \mathbb{C}$. For all functions $f, g \in \mathcal{H}$ we define the function

$$(1.2) \qquad \mathbb{D} \ni z \mapsto F(z; f, g; \nu) := \int_0^z (f'(u))^{|\nu|} e^{\nu g(u)} du\,.$$

2. Main results. In this section we will show our criteria of univalence of the integral operator given by (1.2) and we compare them with results from [2].

Theorem 2.1. *Let $f \in \mathcal{K}$, $g \in \mathcal{H}$ and $|g(z)| \leq M$ for all $z \in \mathbb{D}$ and a certain $M \geq 0$. If*

$$(2.1) \qquad |\nu| \leq \begin{cases} \frac{3\pi}{2(M+\pi)}\,, & \text{for } M < \frac{\pi}{2}\,, \\ \frac{\pi}{2M}\,, & \text{for } M \geq \frac{\pi}{2}\,, \end{cases}$$

then $F(\cdot; f, g; \nu) \in \mathcal{C}$.

Proof. Fix $f \in \mathcal{K}$, $g \in \mathcal{H}$ and $|g(z)| \leq M$ for all $z \in \mathbb{D}$ and a certain $M \geq 0$. We know that $f' \in K(0, 2)$ and by Lemma A we obtain $(f')^{|\nu|} \in K(0, 2|\nu|)$. On the other hand, we get

$$(2.2) \qquad \left|\arg\left(e^{\nu g(z)}\right)\right| = |\operatorname{Im}(\nu g(z))| \leq |\nu g(z)| \leq |\nu| M$$

for $z \in \mathbb{D}$. Consider (1.1) with $f := \mathrm{e}^{\nu g}$. Then for $\mathbb{D} \ni z := r\mathrm{e}^{\mathrm{i}\theta}$ and $0 \le \alpha \le \beta$ we get

$$\arg\left(\mathrm{e}^{\nu g(r\mathrm{e}^{\mathrm{i}\theta_2})}\right) - \arg\left(\mathrm{e}^{\nu g(r\mathrm{e}^{\mathrm{i}\theta_1})}\right) \ge -2|\nu|M \ge -\alpha\pi - \frac{1}{2}(\alpha-\beta)\cdot 0 = -\alpha\pi$$

and for $0 \le \beta < \alpha$ we get

$$\arg\left(\mathrm{e}^{\nu g(r\mathrm{e}^{\mathrm{i}\theta_2})}\right) - \arg\left(\mathrm{e}^{\nu g(r\mathrm{e}^{\mathrm{i}\theta_1})}\right) \ge -2|\nu|M \ge -\alpha\pi - \frac{1}{2}(\alpha-\beta)\cdot(-2\pi) = -\beta\pi.$$

As a consequence $\mathrm{e}^{\nu g} \in K(\alpha,\beta)$ for $\alpha,\beta \ge 2|\nu|M/\pi$. This and Lemma A show that

$$F'(\cdot; f, g; \nu) \in K\left(\frac{2|\nu|M}{\pi}, \frac{2|\nu|(M+\pi)}{\pi}\right).$$

From Lemma B we know that $F'(\cdot; f, g; \nu) \in \mathcal{C}$ if

$$\frac{2|\nu|M}{\pi} \le 1$$

and

$$\frac{2|\nu|(M+\pi)}{\pi} \le 3.$$

Therefore, $F'(\cdot; f, g; \nu) \in \mathcal{C}$ if

$$|\nu| \le \begin{cases} \frac{3\pi}{2(M+\pi)}, & \text{for} \quad M < \frac{\pi}{2}, \\ \frac{\pi}{2M}, & \text{for} \quad M \ge \frac{\pi}{2}. \end{cases}$$

□

Remark 2.2. Let us notice that the assumptions for functions f and g in Theorem 2.1 are much weaker than assumptions in [2, Theorem 2.1]. Since for any $z \in \mathbb{D}$ we have

$$\left|\frac{f''(z)}{f'(z)}\right| \le 1 \Longrightarrow \left|\frac{zf''(z)}{f'(z)}\right| \le |z| < 1 \Longrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \Longrightarrow f \in \mathcal{K},$$

so

$$\left\{f \in \mathcal{H} : \left|\frac{f''(z)}{f'(z)}\right| \le 1 \text{ for every } z \in \mathbb{D}\right\} \subset \mathcal{K}.$$

The inclusion can not be replaced by an equality. For example, the function $\mathbb{D} \ni z \mapsto f(z) := z/(1-z)$ satisfies the condition

$$f \in \mathcal{K} \setminus \left\{f \in \mathcal{H} : \left|\frac{f''(z)}{f'(z)}\right| \le 1 \text{ for every } z \in \mathbb{D}\right\}.$$

Moreover, we assume that g is only bounded while in [2] there is an additional condition for g. For $\nu \in \mathbb{C} \setminus \mathbb{R}$ results are incomparable, since we use $|\nu|$ in the definition of $F(\cdot; f, g; \nu)$. However, in this article there is no additional restriction that $\mathrm{Re}(\nu) \geq 0$. For $\nu > 0$ results from Theorem 2.1 can be directly compared with [2, Theorem 2.1]. For

$$M > \frac{\sqrt{3} + \sqrt{3 + 8\pi^2(\sqrt{3} - 1)}}{4\pi} \approx 0.758$$

results obtained in Theorem 2.1 are better with much weaker assumptions. Moreover, let us point out that in this article we prove that $F(\cdot; f, g; \nu) \in \mathcal{C}$ and not only that $F(\cdot; f, g; \nu) \in \mathcal{S}$.

The following theorem has weaker assumptions about a function f than Theorem 2.1 but still improves results from [2] in some cases.

Theorem 2.3. *Let $f \in \mathcal{C}$, $g \in \mathcal{H}$ and $|g(z)| \leq M$ for all $z \in \mathbb{D}$ and a certain $M \geq 0$. If*

$$|\nu| \leq \frac{\pi}{2M + \pi}\,, \tag{2.3}$$

then $F(\cdot; f, g; \nu) \in \mathcal{C}$.

Proof. Fix $f \in \mathcal{C}$, $g \in \mathcal{H}$ and $|g(z)| \leq M$ for all $z \in \mathbb{D}$ and a certain $M \geq 0$. We know that $f' \in K(1, 3)$ and by Lemma A we obtain $(f')^{|\nu|} \in K(|\nu|, 3|\nu|)$. Analogously to the proof of Theorem 2.1 we get

$$F'(\cdot; f, g; \nu) \in K\left(\frac{|\nu|(2M + \pi)}{\pi}, \frac{|\nu|(2M + 3\pi)}{\pi}\right).$$

From Lemma B we know that $F'(\cdot; f, g; \nu) \in \mathcal{C}$ if

$$\frac{|\nu|(2M + \pi)}{\pi} \leq 1$$

and

$$\frac{|\nu|(2M + 3\pi)}{\pi} \leq 3\,.$$

Therefore, $F'(\cdot; f, g; \nu) \in \mathcal{C}$ if

$$|\nu| \leq \frac{\pi}{2M + \pi}\,.$$

□

Remark 2.4. Let us notice that in Theorem 2.3 we weaken the assumptions for f with respect to Theorem 2.1. Analogously as in Remark 2.2 for $\nu > 0$ results from Theorem 2.3 can be directly compared with [2, Theorem 2.1]. For

$$M > \frac{6\sqrt{3} + \sqrt{108 + 16\pi^2(3\sqrt{3} - 2)}}{8\pi} \approx 1.398$$

the results obtained in Theorem 2.3 are better with much weaker assumptions for f, g and ν. Moreover, let us point out that in this article we prove that $F(\cdot; f, g; \nu) \in \mathcal{C}$ and not only that $F(\cdot; f, g; \nu) \in \mathcal{S}$.

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