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Some properties of the class $\mathcal{U}$

Abstract. In this paper we study the class $\mathcal{U}$ of functions that are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, normalized such that $f(0) = f'(0)-1 = 0$ and satisfy

$$\left| \left[\frac{z}{f(z)} \right]^2 f'(z) - 1 \right| < 1 \quad (z \in \mathbb{D}).$$

For functions in the class $\mathcal{U}$ we give sharp estimates of the second and the third Hankel determinant, its relationship with the class of α-convex functions, as well as certain starlike properties.

1. Introduction. Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}^\star$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ which are starlike and convex in $\mathbb{D}$, respectively, i.e.,

$$\mathcal{S}^\star = \left\{ f \in \mathcal{A} : \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] > 0,\, z \in \mathbb{D} \right\}$$

and

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > 0,\, z \in \mathbb{D} \right\}.$$

Geometrical characterisation of convexity is the usual one, while for the starlikeness we have $f \in \mathcal{S}^\star$, if and only if $f(\mathbb{D})$ is a starlike region, i.e.,

$$z \in f(\mathbb{D}) \quad \Rightarrow \quad tz \in f(\mathbb{D}) \text{ for all } t \in [0, 1].$$

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The linear combination of the expressions involved in the analytical representations of starlikeness and convexity brings us to the classes of α-convex functions introduced in 1969 by Mocanu [3] and consisting of functions $f \in \mathcal{A}$ such that

$$\text{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]\right\} > 0 \quad (z \in \mathbb{D}), \tag{1}$$

where $\frac{f(z)f'(z)}{z} \neq 0$ for $z \in \mathbb{D}$ and $\alpha \in \mathbb{R}$. Those classes he denoted by $\mathcal{M}_\alpha$.

Further, let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying the condition

$$|\text{U}_f(z)| < 1 \quad (z \in \mathbb{D}),$$

where the operator U_f is defined by

$$\text{U}_f(z) := \left[\frac{z}{f(z)}\right]^2 f'(z) - 1.$$

All these classes consist of univalent functions and more details on them can be found in [1, 10].

The class of starlike functions is very large and in the theory of univalent functions it is significant if a class does not entirely lie inside $\mathcal{S}^\star$. One such case is the class of functions with bounded turning consisting of functions f from $\mathcal{A}$ that satisfy $\text{Re} f'(z) > 0$ for all $z \in \mathbb{D}$. Another example is the class $\mathcal{U}$ defined above and first treated in [5] (see also [6, 7, 10]). Namely, the function $-\ln(1-z)$ is convex, thus starlike, but not in $\mathcal{U}$ because $\text{U}_f(0.99) = 3.621\ldots > 1$, while the function f defined by

$$\frac{z}{f(z)} = 1 - \frac{3}{2}z + \frac{1}{2}z^3 = (1-z)^2\left(1 + \frac{z}{2}\right)$$

is in $\mathcal{U}$ and such that

$$\frac{zf'(z)}{f(z)} = -\frac{2\left(z^2+z+1\right)}{z^2+z-2} = -\frac{1}{5} + \frac{3i}{5}$$

for $z = i$. This property is the main reason why the class $\mathcal{U}$ attracts huge attention in the past decades.

In this paper we give sharp estimates of the second and the third Hankel determinant over the class $\mathcal{U}$ and study its relation with the class of α-convex and starlike functions.

2. Main results. In the first theorem we give the sharp estimates of the Hankel determinants of the second and third order for the class $\mathcal{U}$. We first give the definition of the Hankel determinant, whose elements are the coefficients of a function $f \in \mathcal{A}$.

Definition 2. Let $f \in \mathcal{A}$. Then the qth Hankel determinant of f is defined for $q \geq 1$ and $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

Thus, the second and the third Hankel determinants are, respectively,

$$\begin{aligned} &H_2(2) = a_2a_4 - a_3^2, \\ &H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \end{aligned} \tag{3}$$

Theorem 1. *Let $f \in \mathcal{U}$ and $f(z) = z + a_2z^2 + a_3z^3 + \dots$. Then we have the sharp estimates:*

$$|H_2(2)| \leq 1 \quad \textit{and} \quad |H_3(1)| \leq \frac{1}{4}.$$

Proof. In [5] the following characterization of functions f in the class in $\mathcal{U}$ was given:

$$\frac{z}{f(z)} = 1 - a_2z - z\int_0^z \frac{\omega(t)}{t^2}\,dt, \tag{4}$$

where function ω is analytic in $\mathbb{D}$ with $\omega(0) = \omega'(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

If we put $\omega_1(z) = \int_0^z \frac{\omega(t)}{t^2}\,dt$, then we easily obtain $|\omega_1(z)| \leq |z| < 1$ and $|\omega_1'(z)| \leq 1$ for all $z \in \mathbb{D}$. If $\omega_1(z) = c_1z + c_2z^2 + \cdots$, then $\omega_1'(z) = c_1 + 2c_2z + 3c_3z^2 + \cdots$ and $|\omega_1'(z)| \leq 1$, $z \in \mathbb{D}$, gives (see relation (13) in the paper of Prokhorov and Szynal [8]):

(5) $|c_1| \leq 1$, $|2c_2| \leq 1-|c_1|^2$ and $|3c_3(1-|c_1|^2)+4\overline{c_1}c_2^2| \leq (1-|c_1|^2)^2-4|c_2|^2$.

Also, from (4) we have

$$\begin{aligned} f(z) &= \frac{z}{1-(a_2z+c_1z^2+c_2z^3+\cdots)} \\ &= z + a_2z^2 + \left(c_1 + a_2^2\right)z^3 + \left(c_2 + 2a_2c_1 + a_2^3\right)z^4 \\ &\quad + \left(c_3 + 2a_2c_2 + c_1^2 + 3a_2^2c_1 + a_2^4\right)z^5 \cdots . \end{aligned}$$

From the last relation we have

(6) $a_3 = c_1 + a_2^2, \quad a_4 = c_2 + 2a_2c_1 + a_2^3, \quad a_5 = c_3 + 2a_2c_2 + c_1^2 + 3a_2^2c_1 + a_2^4.$

We may suppose that $c_1 \geq 0$, since from (6) we have $c_1 = a_3 - a_2^2$ and a_3 and a_2^2 have the same turn under rotation. In that sense, from (5) we obtain

$$0 \leq c_1 \leq 1, \quad |c_2| \leq \frac{1}{2}\left(1 - c_1^2\right) \quad \text{and} \quad |c_3| \leq \frac{1}{3}\left(1 - c_1^2 - \frac{4|c_2|^2}{1+c_1}\right). \tag{7}$$

If we use (3), (6) and (7), then

$$\begin{aligned}|H_2(2)| &= \left|c_2a_2 - c_1^2\right| \le |c_2|\cdot|a_2| + c_1^2 \le \frac{1}{2}\left(1-c_1^2\right)|a_2| + c_1^2\\ &= \frac{1}{2}\cdot|a_2| + \left(1-\frac{1}{2}\cdot|a_2|\right)c_1^2 \le 1.\end{aligned}$$

The functions $k(z)=\frac{z}{(1-z)^2}$ and $f_1(z)=\frac{z}{1-z^2}$ show that the estimate is the best possible.

Similarly, after some calculations we also have

$$\begin{aligned}|H_3(1)| &= \left|c_1c_3 - c_2^2\right| \le c_1|c_3| + |c_2|^2\\ &\le \frac{1}{3}c_1\left(1-c_1^2-\frac{4|c_2|^2}{1+c_1}\right) + |c_2|^2\\ &= \frac{1}{3}\left(c_1 - c_1^3 + \frac{3-c_1}{1+c_1}|c_2|^2\right)\\ &= \frac{1}{3}\left(c_1 - c_1^3 + \frac{3-c_1}{1+c_1}\cdot\frac{1}{4}\left(1-c_1^2\right)^2\right)\\ &= \frac{1}{12}\left(3-2c_1^2-c_1^4\right) \le \frac{3}{12} = \frac{1}{4}.\end{aligned}$$

The function $f_2(z)=\frac{z}{1-z^3/2}$ shows that the result is the best possible. □

In the rest of the paper we consider some starlikeness problems for the class $\mathcal{U}$ and its connection with the class of α-convex functions.

First, let us recall the classical results about the relation between the starlike functions and α-convex functions.

Theorem 2.

(a) $\mathcal{M}_\alpha \subseteq \mathcal{S}^\star$ *for every real* α *([4]);*

(b) *for* $0 \le \frac{\beta}{\alpha} \le 1$ *we have* $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$ *and for* $\alpha > 1$, $\mathcal{M}_\alpha \subset \mathcal{M}_1 = \mathcal{K}$ *([9, 4]).*

As an analogue of the above theorem we have

Theorem 3. *For the classes* $\mathcal{M}_\alpha$ *the next results are valid.*

(a) $\mathcal{M}_\alpha \subset \mathcal{U}$ *for* $\alpha \le -1$*;*

(b) $\mathcal{M}_\alpha$ *is not a subset of* $\mathcal{U}$ *for any* $0 \le \alpha \le 1$.

Proof. (a) Let $p(z) = \mathrm{U}_f(z)$. Then p is analytic in $\mathbb{D}$ and $p(0)=p'(0)=0$. From this we have $\left[\frac{z}{f(z)}\right]^2 f'(z) = p(z)+1$ and, after some calculations,

$$2\frac{zf'(z)}{f(z)} - \left[1+\frac{zf''(z)}{f'(z)}\right] = 1 - \frac{zp'(z)}{p(z)+1}.$$

The relation (1) is equivalent to

$$\operatorname{Re}\left\{(1+\alpha)\frac{zf'(z)}{f(z)}-\alpha\left[1-\frac{zp'(z)}{p(z)+1}\right]\right\}>0,\ z\in\mathbb{D}. \tag{8}$$

We want to prove that $|p(z)|<1$, $z\in\mathbb{D}$. If not, then according to the Clunie–Jack Lemma ([2]) there exists a z_0, $|z_0|<1$, such that $p(z_0)=e^{i\theta}$ and $z_0p'(z_0)=kp(z_0)=ke^{i\theta}$, $k\geq 2$. For such z_0, from (8) we get

$$\begin{aligned}&\operatorname{Re}\left\{(1+\alpha)\frac{z_0f'(z_0)}{f(z_0)}-\alpha\left[1-\frac{ke^{i\theta}}{e^{i\theta}+1}\right]\right\}\\ &=(1+\alpha)\operatorname{Re}\left[\frac{z_0f'(z_0)}{f(z_0)}\right]+\alpha\frac{k-2}{2}\leq 0\end{aligned}$$

since $f\in\mathcal{S}^\star$ (by Theorem 2) and $\alpha\leq -1$. That is a contradiction to (1). It means that $|p(z)|=|\operatorname{U}_f(z)|<1$, $z\in\mathbb{D}$, i.e., $f\in\mathcal{U}$.

(b) To prove this part, by using Theorem 2(b), it is enough to find a function $g\in\mathcal{K}$ such that g does not belong to the class $\mathcal{U}$. Really, the function $g(z)=-\ln(1-z)$ is convex but not in $\mathcal{U}$. □

Open problem. It remains an open problem to study the relationship between classes $\mathcal{M}_\alpha$ and $\mathcal{U}$ when $-1<\alpha<0$ and $\alpha>1$.

In the next theorem we consider starlikeness of the function

$$g(z)=\frac{z/f(z)-1}{-a_2}, \tag{9}$$

where $f\in\mathcal{U}$ and $a_2=\frac{f''(0)}{2}\neq 0$, i.e., its second coefficient does not vanish.

Namely, we have

Theorem 4. *Let $f\in\mathcal{U}$. Then, for the function g defined by (9) we have:*

(*a*) $|g'(z)-1|<1$ *for* $|z|<|a_2|/2$;

(*b*) $g\in\mathcal{S}^\star$ *in the disk* $|z|<|a_2|/2$ *and even more*

$$\left|\frac{zg'(z)}{g(z)}-1\right|<1 \qquad (|z|<|a_2|/2);$$

(*c*) $g\in\mathcal{U}$ *in the disk* $|z|<|a_2|/2$ *if* $0<|a_2|\leq 1$.

The results are best possible.

Proof. Let $f\in\mathcal{U}$ with $a_2\neq 0$. Then, by using (4), we have

$$\frac{z}{f(z)}=1-a_2z-z\omega_1(z),$$

where ω_1 is analytic in $\mathbb{D}$ such that $|\omega_1(z)|\leq|z|$ and $|\omega_1'(z)|\leq 1$. The appropriate function g from (9) has the form

$$g(z)=z+\frac{1}{a_2}z\omega_1(z).$$

From here $|g'(z)-1|=\frac{1}{|a_2|}|\omega_1(z)+z\omega_1'(z)|<1$ for $|z|<|a_2|/2$.

By using previous representation, we obtain

$$\left|\frac{zg'(z)}{g(z)}-1\right| = \left|\frac{z\omega_1'(z)}{a_2+\omega_1(z)}\right| \le \frac{|z|}{|a_2|-|z|} < 1$$

if $|z|<|a_2|/2$. It means that the function g is starlike in the disk $|z|<|a_2|/2$.

If we consider function f_b defined by

$$\frac{z}{f_b(z)} = 1+bz+z^2, \quad 0<b\le 2, \tag{10}$$

then $f_b\in\mathcal{U}$ and

$$g_b(z) = \frac{\frac{z}{f_b(z)}-1}{b} = z+\frac{1}{b}z^2.$$

For this function we can easily see that for $|z|<b/2$,

$$\mathrm{Re}\frac{zg_b'(z)}{g_b(z)} \ge \frac{1-\frac{2}{b}|z|}{1-\frac{1}{b}|z|} > 0.$$

On the other hand, since $g_b'(-b/2)=0$, the function g_b is not univalent in a bigger disk, which implies that our result is best possible.

Also, by using (9) and the next estimation for the function ω_1:

$$|z\omega_1'(z)-\omega_1(z)| \le \frac{r^2-|\omega_1(z)|^2}{1-r^2},$$

(where $|z|=r$ and $|\omega_1(z)|\le r$), after some calculation, we get

$$\begin{aligned}
|\mathcal{U}_g(z)| &= \left|\frac{\frac{1}{a_2}(z\omega_1'(z)-\omega_1(z))-\frac{1}{a_2^2}\omega_1^2(z)}{\left(1+\frac{1}{a_2}\omega_1(z)\right)^2}\right| \\
&\le \frac{|a_2||z\omega_1'(z)-\omega_1(z)|+|\omega_1(z)|^2}{(|a_2|-|\omega_1(z)|)^2} \\
&\le \frac{|a_2|\frac{r^2-|\omega_1(z)|^2}{1-r^2}+|\omega_1(z)|^2}{(|a_2|-|\omega_1(z)|)^2} \\
&=: \frac{1}{1-r^2}\varphi(t),
\end{aligned}$$

where we put

$$\varphi(t) = \frac{(1-r^2-|a_2|)t^2+|a_2|r^2}{(|a_2|-t)^2}$$

and $|\omega_1(z)|=t$, $0\le t\le r$. We have

$$\begin{aligned}
\varphi'(t) &= \frac{2|a_2|}{(|a_2|-t)^3}\left((1-r^2-|a_2|)t+r^2\right) \\
&= \frac{2|a_2|}{(|a_2|-t)^3}\left((1-|a_2|)t+(1-t)r^2\right) \ge 0,
\end{aligned}$$

because $0 < |a_2| \leq 1$ and $0 \leq t < 1$. It means that φ is an increasing function and

$$\varphi(t) \leq \varphi(r) = \frac{(1-r^2)r^2}{(|a_2|-r)^2}.$$

Finally, we have

$$|\,\mathrm{U}_g(z)| \leq \frac{r^2}{(|a_2|-r)^2} < 1,$$

since $|z| < |a_2|/2$. This implies the second statement of the theorem.

As for sharpness, we can also consider the function f_b defined by (10) with $0 < b \leq 1$. For $|z| < \frac{b}{2}$ we have

$$|\mathrm{U}_{g_b}(z)| \leq \frac{\frac{1}{b^2}|z|^2}{\left(1-\frac{1}{b}|z|\right)^2} < 1,$$

which implies that g_b belongs to the class $\mathcal{U}$ in the disk $|z| < b/2$. □

We believe that part (b) of the previous theorem is valid for all $0 < |a_2| \leq 2$. In that sense we have the next

Conjecture 1. *Let $f \in \mathcal{U}$. Then the function g defined by the expression* (9) *belongs to the class $\mathcal{U}$ in the disk $|z| < |a_2|/2$. The result is the best possible.*

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