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On split r -Jacobsthal quaternions

ABSTRACT. In this paper we introduce a one-parameter generalization of the split Jacobsthal quaternions, namely the split r -Jacobsthal quaternions. We give a generating function, Binet formula for these numbers. Moreover, we obtain some identities, among others Catalan, Cassini identities and convolution identity for the split r -Jacobsthal quaternions.

1. Introduction. A quaternion p is a hyper-complex number represented by an equation

$$p = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where $a, b, c, d \in \mathbb{R}$ and $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal basis in \mathbb{R}^4 , which satisfies the quaternion multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

The quaternions were discovered in 1843 by W. R. Hamilton. In 1849 ([3]), J. Cockle introduced split quaternions, which were called coquaternions. A split quaternion q with real components a_0, a_1, a_2, a_3 and basis $\{1, i, j, k\}$ has the form

$$(1) \quad q = a_0 + a_1i + a_2j + a_3k,$$

where the imaginary units satisfy the non-commutative multiplication rules:

$$(2) \quad i^2 = -1, \quad j^2 = k^2 = ijk = 1,$$

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$$(3) \quad ij = k = -ji, \quad jk = -i = -kj, \quad ki = j = -ik.$$

The scalar and vector parts of a split quaternion $q = a_0 + a_1i + a_2j + a_3k$ are denoted by $S_q = a_0$, $\vec{V}_q = a_1i + a_2j + a_3k$, respectively. Hence we get $q = S_q + \vec{V}_q$. The conjugate of the split quaternion denoted by \bar{q} , is given by

$$\bar{q} = a_0 - a_1i - a_2j - a_3k.$$

The norm of q is defined as

$$(4) \quad N(q) = q\bar{q} = a_0^2 + a_1^2 - a_2^2 - a_3^2.$$

The split quaternions are elements of a 4-dimensional associative algebra. They form a 4-dimensional real vector space equipped with a multiplicative operation. The split quaternions contain nontrivial zero divisors, nilpotent elements and idempotents, for example $\frac{1+j}{2}$ is an idempotent zero divisor, and $i - j$ is nilpotent.

Let q_1, q_2 be any two split quaternions, $q_1 = a_0 + a_1i + a_2j + a_3k$, $q_2 = b_0 + b_1i + b_2j + b_3k$. Then addition and subtraction of the split quaternions are defined as follows:

$$q_1 \pm q_2 = (a_0 \pm b_0) + (a_1 \pm b_1)i + (a_2 \pm b_2)j + (a_3 \pm b_3)k.$$

Multiplication of the split quaternions is defined by

$$\begin{aligned} q_1 \cdot q_2 &= (a_0b_0 - a_1b_1 + a_2b_2 + a_3b_3) + (a_0b_1 + a_1b_0 - a_2b_3 + a_3b_2)i \\ &\quad + (a_0b_2 + a_2b_0 - a_1b_3 + a_3b_1)j + (a_0b_3 + a_3b_0 - a_2b_1 + a_1b_2)k. \end{aligned}$$

2. The r -Jacobsthal numbers. In [6], A. F. Horadam introduced a second order linear recurrence sequence $\{w_n\}$ by the relations

$$(5) \quad w_0 = a, \quad w_1 = b, \quad w_n = pw_{n-1} - qw_{n-2}$$

for $n \geq 2$ and arbitrary integers a, b, p, q . This sequence is a certain generalization of famous sequences such as Fibonacci sequence ($a = 0, b = 1, p = 1, q = -1$), Lucas sequence ($a = 2, b = 1, p = 1, q = -1$), Pell sequence ($a = 0, b = 1, p = 2, q = -1$). Hence sequences defined by (5) are called sequences of the Fibonacci type. Numbers of the Fibonacci type appear in many subjects of mathematics. In [7], A. F. Horadam defined the Fibonacci and Lucas quaternions. In [1], the split Fibonacci quaternions Q_n and the split Lucas quaternions T_n were introduced as follows:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$$

where F_n is the n th Fibonacci number, L_n is the n th Lucas number and i, j, k are split quaternion units which satisfy the rules (2) and (3).

A generalization of the split Fibonacci quaternions split k -Fibonacci quaternions was investigated in [9]. The authors used a generalization of the Fibonacci numbers and the Lucas numbers: k -Fibonacci numbers and

k -Lucas numbers. Some interesting results for the split Pell quaternions and the split Pell–Lucas quaternions can be found in [10]. In [11], the split Jacobsthal quaternions and the split Jacobsthal–Lucas quaternions were considered.

The Jacobsthal sequence $\{J_n\}$ is defined by the recurrence

$$(6) \quad J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2$$

with initial conditions $J_0 = 0$, $J_1 = 1$. The first ten terms of the sequence are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171. This sequence is also given by the Binet-type formula

$$J_n = \frac{2^n - (-1)^n}{3} \text{ for } n \geq 0.$$

Many authors introduced and studied some generalizations of the recurrence of the Jacobsthal sequence, see [4, 5]. The second order recurrence (6) has been generalized in two ways: first, by preserving the initial conditions and second, by preserving the recurrence relation. In [2], a one-parameter generalization of the Jacobsthal numbers was introduced. We recall this generalization.

Let $n \geq 0$, $r \geq 0$ be integers. The n th r -Jacobsthal number $J(r, n)$ is defined as follows:

$$(7) \quad J(r, n) = 2^r J(r, n-1) + (2^r + 4^r) J(r, n-2) \text{ for } n \geq 2$$

with $J(r, 0) = 1$, $J(r, 1) = 1 + 2^{r+1}$.

For $r = 0$ we have $J(0, n) = J_{n+2}$. By (7) we obtain

$$(8) \quad \begin{aligned} J(r, 0) &= 1 \\ J(r, 1) &= 2 \cdot 2^r + 1 \\ J(r, 2) &= 3 \cdot 4^r + 2 \cdot 2^r \\ J(r, 3) &= 5 \cdot 8^r + 5 \cdot 4^r + 2^r \\ J(r, 4) &= 8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r \\ J(r, 5) &= 13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r. \end{aligned}$$

In [2], it was proved that the r -Jacobsthal numbers can be used for counting of independent sets of special classes of graphs. We will recall some properties of the r -Jacobsthal numbers.

Theorem 1 ([2], Binet formula). *For $n \geq 0$, the n th r -Jacobsthal number is given by*

$$J(r, n) = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} \lambda_1^n + \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} \lambda_2^n,$$

where

$$\lambda_1 = 2^{r-1} + \frac{1}{2} \sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \quad \lambda_2 = 2^{r-1} - \frac{1}{2} \sqrt{4 \cdot 2^r + 5 \cdot 4^r}.$$

Theorem 2 ([2]). *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$(9) \quad \sum_{l=0}^{n-1} J(r, l) = \frac{J(r, n) + (2^r + 4^r)J(r, n-1) - 2 - 2^r}{4^r + 2^{r+1} - 1}.$$

Theorem 3 ([2], Cassini identity). *Let $n \geq 1$. Then*

$$J(r, n+1)J(r, n-1) - J^2(r, n) = (-1)^n (2^r + 1)^2 (2^r + 4^r)^{n-1}.$$

Theorem 4 ([2], convolution identity). *Let n, m, r be integers such that $m \geq 2$, $n \geq 1$, $r \geq 0$. Then*

$$J(r, m+n) = 2^r J(r, m-1)J(r, n) + (4^r + 8^r)J(r, m-2)J(r, n-1).$$

In this paper, we introduce and study split r -Jacobsthal quaternions. Another generalization of the split Jacobsthal quaternions was studied in [8].

3. Some properties of the split r -Jacobsthal quaternions. For $n \geq 0$, the split r -Jacobsthal quaternion JSQ_n^r we define by

$$(10) \quad JSQ_n^r = J(r, n) + iJ(r, n+1) + jJ(r, n+2) + kJ(r, n+3),$$

where $J(r, n)$ is the n th r -Jacobsthal number, defined by (7) and i, j, k are split quaternion units which satisfy the multiplication rules (2) and (3).

By (8) and (10) we obtain

$$(11) \quad \begin{aligned} JSQ_0^r &= 1 + i(2^{r+1} + 1) + j(3 \cdot 4^r + 2^{r+1}) + k(5 \cdot 8^r + 5 \cdot 4^r + 2^r) \\ JSQ_1^r &= 2^{r+1} + 1 + i(3 \cdot 4^r + 2^{r+1}) + j(5 \cdot 8^r + 5 \cdot 4^r + 2^r) \\ &\quad + k(8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r) \\ JSQ_2^r &= 3 \cdot 4^r + 2^{r+1} + i(5 \cdot 8^r + 5 \cdot 4^r + 2^r) \\ &\quad + j(8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r) \\ &\quad + k(13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r). \end{aligned}$$

Using the formula $J(0, n) = J_{n+2}$, we obtain $JSQ_n^0 = JSQ_{n+2}$, where JSQ_n is the n th split Jacobsthal quaternion introduced in [11].

Proposition 5. *Let $n \geq 0$, $r \geq 0$. Then*

$$\begin{aligned} N(JSQ_n^r) &= (1 - 4^r - 2 \cdot 8^r - 2 \cdot 16^r - 2 \cdot 32^r - 64^r)J^2(r, n) \\ &\quad + (1 - 2 \cdot 4^r - 4 \cdot 8^r - 4 \cdot 16^r)J^2(r, n+1) \\ &\quad - 2(4^r + 2 \cdot 8^r + 3 \cdot 16^r + 2 \cdot 32^r)J(r, n)J(r, n+1). \end{aligned}$$

Proof. By (7) we get

$$\begin{aligned} J(r, n+2) &= 2^r J(r, n+1) + (2^r + 4^r)J(r, n), \\ J(r, n+3) &= (2^r + 2 \cdot 4^r)J(r, n+1) + (4^r + 8^r)J(r, n). \end{aligned}$$

Let $A = J(r, n + 1)$, $B = J(r, n)$. Using formula (4), we obtain

$$\begin{aligned} N(JSQ_n^r) &= A^2 + B^2 - (2^r A + (2^r + 4^r)B)^2 - ((2^r + 2 \cdot 4^r)A \\ &\quad + (4^r + 8^r)B)^2 \\ &= [1 - 4^r - (2 \cdot 4^r + 2^r)^2]A^2 + [1 - (2^r + 4^r)^2 - (4^r + 8^r)^2]B^2 \\ &\quad - 2[4^r + 8^r + (2 \cdot 4^r + 2^r)(4^r + 8^r)]AB. \end{aligned}$$

By simple calculations we get the result. \square

Proposition 6. *Let $n \geq 2$, $r \geq 0$. Then*

$$JSQ_n^r = 2^r JSQ_{n-1}^r + (2^r + 4^r)JSQ_{n-2}^r,$$

where JSQ_0^r, JSQ_1^r are given in (11).

Proof. By (10) we get

$$\begin{aligned} &2^r JSQ_{n-1}^r + (2^r + 4^r)JSQ_{n-2}^r \\ &= 2^r (J(r, n - 1) + iJ(r, n) + jJ(r, n + 1) + kJ(r, n + 2)) \\ &\quad + (2^r + 4^r)(J(r, n - 2) + iJ(r, n - 1) + jJ(r, n) + kJ(r, n + 1)) \\ &= J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3) = JSQ_n^r. \quad \square \end{aligned}$$

Proposition 7. *Let $n \geq 0$, $r \geq 0$. Then*

- (i) $JSQ_n^r + \overline{JSQ_n^r} = 2J(r, n)$,
- (ii) $(JSQ_n^r)^2 = 2J(r, n)JSQ_n^r - N(JSQ_n^r)$,
- (iii) $JSQ_n^r - iJSQ_{n+1}^r - jJSQ_{n+2}^r - kJSQ_{n+3}^r$
 $= J(r, n) + J(r, n + 2) - J(r, n + 4) - J(r, n + 6)$.

Proof. (i) By the definition of the conjugate of the split quaternion we have

$$\begin{aligned} JSQ_n^r + \overline{JSQ_n^r} &= J(r, n) + iJ(r, n + 1) + jJ(r, n + 2) + kJ(r, n + 3) \\ &\quad + J(r, n) - iJ(r, n + 1) - jJ(r, n + 2) - kJ(r, n + 3) \\ &= 2J(r, n). \end{aligned}$$

(ii) By simple calculations we obtain

$$\begin{aligned} (JSQ_n^r)^2 &= J^2(r, n) - J^2(r, n + 1) + J^2(r, n + 2) + J^2(r, n + 3) \\ &\quad + 2(iJ(r, n)J(r, n + 1) + jJ(r, n)J(r, n + 2) + J(r, n)J(r, n + 3)) \\ &\quad + (ij + ji)J(r, n + 1)J(r, n + 2) + (ik + ki)J(r, n + 1)J(r, n + 3) \\ &\quad + (jk + kj)J(r, n + 2)J(r, n + 3). \end{aligned}$$

By (3) we get

$$\begin{aligned}
(JSQ_n^r)^2 &= -J^2(r, n) - J^2(r, n+1) + J^2(r, n+2) + J^2(r, n+3) \\
&\quad + 2(J^2(r, n) + iJ(r, n)J(r, n+1) \\
&\quad + jJ(r, n)J(r, n+2) + kJ(r, n)J(r, n+3)) \\
&= 2J(r, n)(J(r, n) + iJ(r, n+1) + jJ(r, n+2) + kJ(r, n+3)) \\
&\quad - (J^2(r, n) + J^2(r, n+1) - J^2(r, n+2) - J^2(r, n+3)) \\
&= 2J(r, n)JSQ_n^r - N(JSQ_n^r).
\end{aligned}$$

(iii)

$$\begin{aligned}
&JSQ_n^r - iJSQ_{n+1}^r - jJSQ_{n+2}^r - kJSQ_{n+3}^r \\
&= J(r, n) + iJ(r, n+1) + jJ(r, n+2) + kJ(r, n+3) \\
&\quad - i(J(r, n+1) + iJ(r, n+2) + jJ(r, n+3) + kJ(r, n+4)) \\
&\quad - j(J(r, n+2) + iJ(r, n+3) + jJ(r, n+4) + kJ(r, n+5)) \\
&\quad - k(J(r, n+3) + iJ(r, n+4) + jJ(r, n+5) + kJ(r, n+6)) \\
&= J(r, n) + J(r, n+2) - J(r, n+4) - J(r, n+6) \\
&\quad - (ij + ji)J(r, n+3) - (ik + ki)J(r, n+4) - (jk + kj)J(r, n+5).
\end{aligned}$$

Using equalities $ij + ji = 0$, $ik + ki = 0$ and $jk + kj = 0$, we get

$$\begin{aligned}
&JSQ_n^r - iJSQ_{n+1}^r - jJSQ_{n+2}^r - kJSQ_{n+3}^r \\
&= J(r, n) + J(r, n+2) - J(r, n+4) - J(r, n+6). \quad \square
\end{aligned}$$

Now we present the Binet formula for the split r -Jacobsthal quaternions.

Theorem 8 (Binet formula). *Let $n \geq 0$, $r \geq 0$. Then*

$$(12) \quad JSQ_n^r = C_1 \underline{\alpha} \alpha^n + C_2 \underline{\beta} \beta^n,$$

where

$$\begin{aligned}
\alpha &= 2^{r-1} + \frac{1}{2} \sqrt{4 \cdot 2^r + 5 \cdot 4^r}, & \beta &= 2^{r-1} - \frac{1}{2} \sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \\
\underline{\alpha} &= 1 + i\alpha + j\alpha^2 + k\alpha^3, & \underline{\beta} &= 1 + i\beta + j\beta^2 + k\beta^3, \\
C_1 &= \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}, & C_2 &= \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}.
\end{aligned}$$

Proof. By the Binet formula for the r -Jacobsthal numbers we obtain

$$\begin{aligned}
JSQ_n^r &= J(r, n) + iJ(r, n+1) + jJ(r, n+2) + kJ(r, n+3) \\
&= C_1 \alpha^n + C_2 \beta^n + i(C_1 \alpha^{n+1} + C_2 \beta^{n+1}) \\
&\quad + j(C_1 \alpha^{n+2} + C_2 \beta^{n+2}) + k(C_1 \alpha^{n+3} + C_2 \beta^{n+3}) \\
&= C_1 \alpha^n (1 + i\alpha + j\alpha^2 + k\alpha^3) + C_2 \beta^n (1 + i\beta + j\beta^2 + k\beta^3) \\
&= C_1 \underline{\alpha} \alpha^n + C_2 \underline{\beta} \beta^n. \quad \square
\end{aligned}$$

In particular, we obtain the Binet formula for the split Jacobsthal quaternions (see [11]).

Corollary 9. *Let $n \geq 0$ be an integer. Then*

$$JSQ_n = \frac{1}{3} [2^n(1 + 2i + 4j + 8k) - (-1)^n(1 - i + j - k)].$$

Proof. By Theorem 8, for $r = 0$ we have $C_1 = \frac{4}{3}$, $C_2 = -\frac{1}{3}$, $\alpha = 2$, $\beta = -1$ and

$$\begin{aligned} JSQ_n^0 &= \frac{4}{3} \cdot 2^n(1 + 2i + 4j + 8k) - \frac{1}{3}(-1)^n(1 - i + j - k) \\ &= \frac{1}{3} \cdot 2^{n+2}(1 + 2i + 4j + 8k) - \frac{1}{3}(-1)^{n+2}(1 - i + j - k) = JSQ_{n+2}. \quad \square \end{aligned}$$

The Binet formula (12) can be used for proving some identities for the split r -Jacobsthal quaternions. We will need the following lemma.

Lemma 10. *Let*

$$\begin{aligned} \alpha &= 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \\ \beta &= 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \\ \underline{\alpha} &= 1 + i\alpha + j\alpha^2 + k\alpha^3, \\ \underline{\beta} &= 1 + i\beta + j\beta^2 + k\beta^3. \end{aligned}$$

Then

$$\begin{aligned} \underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha} &= 2[1 + 4^r + 2^r + (4^r + 2^r)^2 - (4^r + 2^r)^3 \\ &\quad + 2^r i + (3 \cdot 4^r + 2^{r+1})j + (4 \cdot 8^r + 3 \cdot 4^r)k]. \end{aligned}$$

Proof. By (2) and (3) we have

$$\begin{aligned} \underline{\alpha}\underline{\beta} &= 1 - \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3 + i(\alpha + \beta + (\alpha\beta)^2(\alpha - \beta)) \\ &\quad + j(\alpha^2 + \beta^2 + \alpha\beta(\alpha^2 - \beta^2)) + k(\alpha^3 + \beta^3 + \alpha\beta(\beta - \alpha)), \\ \underline{\beta}\underline{\alpha} &= 1 - \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3 + i(\alpha + \beta - (\alpha\beta)^2(\alpha - \beta)) \\ &\quad + j(\alpha^2 + \beta^2 - \alpha\beta(\alpha^2 - \beta^2)) + k(\alpha^3 + \beta^3 - \alpha\beta(\beta - \alpha)). \end{aligned}$$

Hence we obtain

$$\begin{aligned} \underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha} &= 2[1 - \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3 + i(\alpha + \beta) \\ &\quad + j(\alpha^2 + \beta^2) + k(\alpha^3 + \beta^3)]. \end{aligned}$$

Note that

$$\begin{aligned} \alpha + \beta &= 2^r, \\ \alpha - \beta &= \sqrt{4 \cdot 2^r + 5 \cdot 4^r}, \\ \alpha\beta &= -(4^r + 2^r), \end{aligned}$$

$$\begin{aligned}\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta = 3 \cdot 4^r + 2^{r+1}, \\ \alpha^3 + \beta^3 &= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = 4 \cdot 8^r + 3 \cdot 4^r.\end{aligned}$$

Hence we get

$$(13) \quad \underline{\alpha\beta} + \underline{\beta\alpha} = 2[1 + 4^r + 2^r + (4^r + 2^r)^2 - (4^r + 2^r)^3 + 2^r i + (3 \cdot 4^r + 2^{r+1})j + (4 \cdot 8^r + 3 \cdot 4^r)k].$$

Moreover,

$$(14) \quad \begin{aligned}\underline{\alpha\beta} &= 1 + 4^r + 2^r + (4^r + 2^r)^2 - (4^r + 2^r)^3 \\ &\quad + i(2^r + (4^r + 2^r)^2 \sqrt{4 \cdot 2^r + 5 \cdot 4^r}) \\ &\quad + j(3 \cdot 4^r + 2^{r+1} - (8^r + 4^r) \sqrt{4 \cdot 2^r + 5 \cdot 4^r}) \\ &\quad + k(4 \cdot 8^r + 3 \cdot 4^r + (4^r + 2^r) \sqrt{4 \cdot 2^r + 5 \cdot 4^r}),\end{aligned}$$

$$(15) \quad \begin{aligned}\underline{\beta\alpha} &= 1 + 4^r + 2^r + (4^r + 2^r)^2 - (4^r + 2^r)^3 \\ &\quad + i(2^r - (4^r + 2^r)^2 \sqrt{4 \cdot 2^r + 5 \cdot 4^r}) \\ &\quad + j(3 \cdot 4^r + 2^{r+1} + (8^r + 4^r) \sqrt{4 \cdot 2^r + 5 \cdot 4^r}) \\ &\quad + k(4 \cdot 8^r + 3 \cdot 4^r - (4^r + 2^r) \sqrt{4 \cdot 2^r + 5 \cdot 4^r}). \quad \square\end{aligned}$$

Now we will give some identities such as Catalan, Cassini and d'Ocagne identities for the split r -Jacobsthal quaternions.

Theorem 11 (Catalan identity). *Let $n \geq 0$, $r \geq 0$ be integers such that $m \geq n$. Then*

$$\begin{aligned}(JSQ_m^r)^2 - JSQ_{m+n}^r JSQ_{m-n}^r \\ = -\frac{(1+2^r)^2(-4^r-2^r)^{m-n}}{4 \cdot 2^r + 5 \cdot 4^r} ((-4^r-2^r)^n(\underline{\alpha\beta} + \underline{\beta\alpha}) - \alpha^{2n}\underline{\alpha\beta} - \beta^{2n}\underline{\beta\alpha}).\end{aligned}$$

Proof. By formula (12) we get

$$\begin{aligned}(JSQ_m^r)^2 - JSQ_{m+n}^r JSQ_{m-n}^r \\ = (C_1\underline{\alpha}\alpha^m + C_2\underline{\beta}\beta^m)(C_1\underline{\alpha}\alpha^m + C_2\underline{\beta}\beta^m) \\ - (C_1\underline{\alpha}\alpha^{m+n} + C_2\underline{\beta}\beta^{m+n})(C_1\underline{\alpha}\alpha^{m-n} + C_2\underline{\beta}\beta^{m-n}) \\ = C_1C_2[(\alpha\beta)^m(\underline{\alpha\beta} + \underline{\beta\alpha}) - (\alpha\beta)^{m-n}(\alpha^{2n}\underline{\alpha\beta} + \beta^{2n}\underline{\beta\alpha})].\end{aligned}$$

Using the formula $\alpha\beta = -(4^r + 2^r)$, we obtain

$$\begin{aligned}(JSQ_m^r)^2 - JSQ_{m+n}^r JSQ_{m-n}^r \\ = C_1C_2(-4^r - 2^r)^{m-n} ((-4^r - 2^r)^n(\underline{\alpha\beta} + \underline{\beta\alpha}) - \alpha^{2n}\underline{\alpha\beta} - \beta^{2n}\underline{\beta\alpha}),\end{aligned}$$

where

$$C_1C_2 = -\frac{(1+2^r)^2}{4 \cdot 2^r + 5 \cdot 4^r}$$

and $\underline{\alpha\beta} + \underline{\beta\alpha}$, $\underline{\alpha\beta}$, $\underline{\beta\alpha}$ are given by (13), (14), (15), respectively. \square

Note that for $n = 1$ we get the Cassini identity for the split r -Jacobsthal quaternions.

Corollary 12. *For $m \geq 1, r \geq 0$ we have*

$$\begin{aligned} & (JSQ_m^r)^2 - JSQ_{m+1}^r JSQ_{m-1}^r \\ &= -\frac{(1+2^r)^2(-4^r-2^r)^{m-1}}{4 \cdot 2^r + 5 \cdot 4^r} \left(-(4^r+2^r)(\underline{\alpha}\underline{\beta} + \underline{\beta}\underline{\alpha}) - \alpha^2\underline{\alpha}\underline{\beta} - \beta^2\underline{\beta}\underline{\alpha} \right). \end{aligned}$$

In particular, we obtain the Cassini identity for the split Jacobsthal quaternions (see [11]).

Corollary 13. *Let $m \geq 1$ be an integer. Then*

$$(JSQ_m)^2 - JSQ_{m+1} JSQ_{m-1} = (-2)^{m-1}(-1 + 5i + 3j + 9k).$$

Proof. By (14) and (15) for $r = 0$ we have

$$\begin{aligned} \underline{\alpha}\underline{\beta} &= -1 + 13i - j + 13k, \\ \underline{\beta}\underline{\alpha} &= -1 - 11i + 11j + k. \end{aligned}$$

By Corollary 12 we get

$$\begin{aligned} & (JSQ_m^0)^2 - JSQ_{m+1}^0 JSQ_{m-1}^0 \\ &= -\frac{4(-2)^{m-1}}{9} \left(-2(-2 + 2i + 10j + 14k) \right. \\ &\quad \left. - 4(-1 + 13i - j + 13k) - (-1 - 11i + 11j + k) \right) \\ &= 4(-2)^{m-1}(-1 + 5i + 3j + 9k). \end{aligned}$$

Using the formula $JSQ_m^0 = JSQ_{m+2}$, we get the result. \square

Theorem 14 (d'Ocagne identity). *Let m, n, r be integers. Then*

$$\begin{aligned} & JSQ_n^r JSQ_{m+1}^r - JSQ_{n+1}^r JSQ_m^r \\ &= \frac{(1+2^r)^2 \sqrt{4 \cdot 2^r + 5 \cdot 4^r}}{4 \cdot 2^r + 5 \cdot 4^r} (-4^r - 2^r)^m (\alpha^{n-m} \underline{\alpha}\underline{\beta} - \beta^{n-m} \underline{\beta}\underline{\alpha}), \end{aligned}$$

where $\underline{\alpha}\underline{\beta}, \underline{\beta}\underline{\alpha}$ are given by (14), (15), respectively.

Proof. By formula (12) we get

$$\begin{aligned} & JSQ_n^r JSQ_{m+1}^r - JSQ_{n+1}^r JSQ_m^r \\ &= (C_1 \underline{\alpha} \alpha^n + C_2 \underline{\beta} \beta^n)(C_1 \underline{\alpha} \alpha^{m+1} + C_2 \underline{\beta} \beta^{m+1}) \\ &\quad - (C_1 \underline{\alpha} \alpha^{n+1} + C_2 \underline{\beta} \beta^{n+1})(C_1 \underline{\alpha} \alpha^m + C_2 \underline{\beta} \beta^m) \\ &= C_1 C_2 (\beta - \alpha) (\alpha^n \beta^m \underline{\alpha}\underline{\beta} - \alpha^m \beta^n \underline{\beta}\underline{\alpha}) \\ &= C_1 C_2 (\beta - \alpha) (\alpha\beta)^m (\alpha^{n-m} \underline{\alpha}\underline{\beta} - \beta^{n-m} \underline{\beta}\underline{\alpha}) \\ &= \frac{(1+2^r)^2 \sqrt{4 \cdot 2^r + 5 \cdot 4^r}}{4 \cdot 2^r + 5 \cdot 4^r} (-4^r - 2^r)^m (\alpha^{n-m} \underline{\alpha}\underline{\beta} - \beta^{n-m} \underline{\beta}\underline{\alpha}). \quad \square \end{aligned}$$

In the next theorem we give a summation formula for the split r -Jacobsthal quaternions.

Theorem 15. *Let $n \geq 1$, $r \geq 0$. Then*

$$\sum_{l=0}^n JSQ_l^r = \frac{JSQ_{n+1}^r + (2^r + 4^r)JSQ_n^r - (1 + i + j + k)(2 + 2^r)}{4^r + 2^{r+1} - 1} \\ - i - j(2 + 2^{r+1}) - k(2^{r+2} + 3 \cdot 4^r + 2).$$

Proof. Using Theorem 2, we get

$$\begin{aligned} \sum_{l=0}^n JSQ_l^r &= \sum_{l=0}^n (J(r, l) + iJ(r, l+1) + jJ(r, l+2) + kJ(r, l+3)) \\ &= \sum_{l=0}^n J(r, l) + i \sum_{l=0}^n J(r, l+1) + j \sum_{l=0}^n J(r, l+2) + k \sum_{l=0}^n J(r, l+3) \\ &= \frac{1}{4^r + 2^{r+1} - 1} [J(r, n+1) + (2^r + 4^r)J(r, n) - 2 - 2^r \\ &\quad + i(J(r, n+2) + (2^r + 4^r)J(r, n+1) - 2 - 2^r - J(r, 0)) \\ &\quad + j(J(r, n+3) + (2^r + 4^r)J(r, n+2) - 2 - 2^r - J(r, 0) - J(r, 1)) \\ &\quad + k(J(r, n+4) + (2^r + 4^r)J(r, n+3) - 2 - 2^r \\ &\quad - J(r, 0) - J(r, 1) - J(r, 2))]. \end{aligned}$$

By simple calculations we obtain

$$\begin{aligned} \sum_{l=0}^n JSQ_l^r &= \frac{1}{4^r + 2^{r+1} - 1} [J(r, n+1) + iJ(r, n+2) \\ &\quad + jJ(r, n+3) + kJ(r, n+4) \\ &\quad + (2^r + 4^r)(J(r, n) + iJ(r, n+1) + jJ(r, n+2) + kJ(r, n+3)) \\ &\quad - (2 + 2^r)(1 + i + j + k)] - i - j(2^{r+1} + 2) - k(2^{r+2} + 3 \cdot 4^r + 2) \\ &= \frac{JSQ_{n+1}^r + (2^r + 4^r)JSQ_n^r - (1 + i + j + k)(2 + 2^r)}{4^r + 2^{r+1} - 1} \\ &\quad - i - j(2 + 2^{r+1}) - k(2^{r+2} + 3 \cdot 4^r + 2). \quad \square \end{aligned}$$

The next theorem gives the convolution identity for the split r -Jacobsthal quaternions.

Theorem 16. *Let $m \geq 2$, $n \geq 1$, $r \geq 0$. Then*

$$\begin{aligned} 2JSQ_{m+n}^r &= 2^r JSQ_{m-1}^r JSQ_n^r + (4^r + 8^r)JSQ_{m-2}^r JSQ_{n-1}^r \\ &\quad + J(r, m+n) + J(r, m+n+2) - J(r, m+n+4) - J(r, m+n+6). \end{aligned}$$

Proof. By simple calculations we have

$$\begin{aligned}
& 2^r JSQ_{m-1}^r JSQ_n^r \\
&= 2^r (J(r, m-1)J(r, n) + iJ(r, m-1)J(r, n+1) \\
&+ jJ(r, m-1)J(r, n+2) + kJ(r, m-1)J(r, n+3) \\
&+ iJ(r, m)J(r, n) - J(r, m)J(r, n+1) + kJ(r, m)J(r, n+2) \\
&- jJ(r, m)J(r, n+3) + jJ(r, m+1)J(r, n) - kJ(r, m+1)J(r, n+1) \\
&+ J(r, m+1)J(r, n+2) - iJ(r, m+1)J(r, n+3) + kJ(r, m+2)J(r, n) \\
&+ jJ(r, m+2)J(r, n+1) + iJ(r, m+2)J(r, n+2) \\
&+ J(r, m+2)J(r, n+3)).
\end{aligned}$$

Moreover,

$$\begin{aligned}
& (4^r + 8^r) JSQ_{m-2}^r JSQ_{n-1}^r \\
&= (4^r + 8^r) (J(r, m-2)J(r, n-1) + iJ(r, m-2)J(r, n) \\
&+ jJ(r, m-2)J(r, n+1) + kJ(r, m-2)J(r, n+2) \\
&+ iJ(r, m-1)J(r, n-1) \\
&- J(r, m-1)J(r, n) + kJ(r, m-1)J(r, n+1) - jJ(r, m-1)J(r, n+2) \\
&+ jJ(r, m)J(r, n-1) - kJ(r, m)J(r, n) + J(r, m)J(r, n+1) \\
&- iJ(r, m)J(r, n+2) + kJ(r, m+1)J(r, n-1) + jJ(r, m+1)J(r, n) \\
&+ iJ(r, m+1)J(r, n+1) + J(r, m+1)J(r, n+2)).
\end{aligned}$$

Hence

$$\begin{aligned}
& 2^r JSQ_{m-1}^r JSQ_n^r + (4^r + 8^r) JSQ_{m-2}^r JSQ_{n-1}^r \\
&= 2^r J(r, m-1)J(r, n) + (4^r + 8^r) (J(r, m-2)J(r, n-1) \\
&+ i(2^r J(r, m-1)J(r, n+1) + (4^r + 8^r)J(r, m-2)J(r, n)) \\
&+ j(2^r J(r, m-1)J(r, n+2) + (4^r + 8^r)J(r, m-2)J(r, n+1)) \\
&+ k(2^r J(r, m-1)J(r, n+3) + (4^r + 8^r)J(r, m-2)J(r, n+2)) \\
&+ i(2^r J(r, m)J(r, n) + (4^r + 8^r)J(r, m-1)J(r, n-1)) \\
&+ j(2^r J(r, m+1)J(r, n) + (4^r + 8^r)J(r, m)J(r, n-1)) \\
&+ k(2^r J(r, m)J(r, n+2) + (4^r + 8^r)J(r, m-1)J(r, n+1)) \\
&- 2^r J(r, m)J(r, n+1) - (4^r + 8^r)J(r, m-1)J(r, n) \\
&+ 2^r J(r, m+1)J(r, n+2) - (4^r + 8^r)J(r, m)J(r, n+1) \\
&+ 2^r J(r, m+2)J(r, n+3) - (4^r + 8^r)J(r, m+1)J(r, n+2) \\
&+ i[2^r J(r, m+2)J(r, n+2) + (4^r + 8^r)J(r, m+1)J(r, n+1) \\
&- 2^r J(r, m+1)J(r, n+3) - (4^r + 8^r)J(r, m)J(r, n+2)]
\end{aligned}$$

$$\begin{aligned}
& + j[2^r J(r, m+2)J(r, n+1) + (4^r + 8^r)J(r, m+1)J(r, n) \\
& \quad - 2^r J(r, m)J(r, n+3) - (4^r + 8^r)J(r, m-1)J(r, n+2)] \\
& + k[2^r J(r, m+2)J(r, n) + (4^r + 8^r)J(r, m+1)J(r, n-1) \\
& \quad - 2^r J(r, m+1)J(r, n+1) - (4^r + 8^r)J(r, m)J(r, n)].
\end{aligned}$$

Using Theorem 4, we get

$$\begin{aligned}
& 2^r JSQ_{m-1}^r JSQ_n^r + (4^r + 8^r)JSQ_{m-2}^r JSQ_{n-1}^r \\
& = J(r, m+n) + 2(iJ(r, m+n+1) + jJ(r, m+n+2) \\
& \quad + kJ(r, m+n+3)) - J(r, m+n+2) + J(r, m+n+4) \\
& \quad + J(r, m+n+6) \\
& = -J(r, m+n) - J(r, m+n+2) + J(r, m+n+4) + J(r, m+n+6) \\
& \quad + 2(J(r, m+n) + iJ(r, m+n+1) + jJ(r, m+n+2) \\
& \quad + kJ(r, m+n+3)) \\
& = 2JSQ_{m+n}^r - (J(r, m+n) + J(r, m+n+2) \\
& \quad - J(r, m+n+4) - J(r, m+n+6)).
\end{aligned}$$

Hence we get the result. \square

Now we will give the generating function for the split r -Jacobsthal quaternion sequence. Similarly as the Jacobsthal sequence, r -Jacobsthal sequence, this sequence can be considered as the coefficients of the power series expansion of the corresponding generating function. We recall the result for the r -Jacobsthal sequence.

Theorem 17 ([2]). *The generating function of the sequence of r -Jacobsthal numbers has the following form:*

$$f(t) = \frac{1 + (1 + 2^r)t}{1 - 2^r t - (2^r + 4^r)t^2}.$$

Theorem 18. *The generating function for the split r -Jacobsthal quaternion sequence $\{JSQ_n^r\}$ has the following form:*

$$g(t) = \frac{JSQ_0^r + (JSQ_1^r - 2^r JSQ_0^r)t}{1 - 2^r t - (2^r + 4^r)t^2}.$$

Proof. Let

$$g(t) = JSQ_0^r + JSQ_1^r t + JSQ_2^r t^2 + \dots + JSQ_n^r t^n + \dots$$

be the generating function of the split r -Jacobsthal quaternion sequence. Then

$$\begin{aligned}
2^r t g(t) & = 2^r JSQ_0^r t + 2^r JSQ_1^r t^2 + 2^r JSQ_2^r t^3 + \dots \\
& \quad + 2^r JSQ_{n-1}^r t^n + \dots,
\end{aligned}$$

$$(2^r + 4^r)t^2g(t) = (2^r + 4^r)JSQ_0^r t^2 + (2^r + 4^r)JSQ_1^r t^3 \\ + (2^r + 4^r)JSQ_2^r t^4 + \cdots + (2^r + 4^r)JSQ_{n-2}^r t^n + \cdots .$$

By Proposition 6 we get

$$g(t) - 2^r t g(t) - (2^r + 4^r)t^2 g(t) \\ = JSQ_0^r + (JSQ_1^r - 2^r JSQ_0^r)t + (JSQ_2^r - 2^r JSQ_1^r \\ - (2^r + 4^r)JSQ_0^r)t^2 + \cdots \\ = JSQ_0^r + (JSQ_1^r - 2^r JSQ_0^r)t.$$

Thus

$$g(t) = \frac{JSQ_0^r + (JSQ_1^r - 2^r JSQ_0^r)t}{1 - 2^r t - (2^r + 4^r)t^2}.$$

Using equalities (11), we obtain

$$JSQ_0^r = 1 + i(2^{r+1} + 1) + j(3 \cdot 4^r + 2^{r+1}) \\ + k(5 \cdot 8^r + 5 \cdot 4^r + 2^r), \\ JSQ_1^r - 2^r JSQ_0^r = 2^r + 1 + i(4^r + 2^r) + j(2 \cdot 8^r + 3 \cdot 4^r + 2^r) \\ + k(3 \cdot 16^r + 5 \cdot 8^r + 2 \cdot 4^r). \quad \square$$

4. Conclusion. In this study, a one-parameter generalization of the split Jacobsthal quaternions was introduced. Some results including the Binet formula, generating function, a summation formula for these quaternions were given. Moreover, some identities, such as Catalan, Cassini, d'Ocagne and convolution identities, involving the split r -Jacobsthal quaternions were obtained. The presented results are generalization of the known results for the split Jacobsthal quaternions.

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