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On extensions of matrix-valued Hahn–Sturm–Liouville operators

Abstract. In this paper, we study matrix-valued Hahn–Sturm–Liouville equations. We give an existence and uniqueness result. We introduce the corresponding maximal and minimal operators for this system, and some properties of these operators are investigated. Finally, we characterize extensions (maximal dissipative, maximal accumulative and self-adjoint) of the minimal symmetric operator.

1. Introduction. As is known, extension theory of symmetric operators is one of the main research areas of operator theory. This theory was studied earlier [33]. In [17], the description of self-adjoint extensions of a symmetric operator was given. Rofe-Beketov [31] obtained extensions of a symmetric operator with aid of linear relations. Later, in [16, 24], the notion of a space of boundary values was introduced. In [26], a description of extensions of a second-order symmetric operator was given. In [19], the author obtained a description of self-adjoint extensions of Sturm–Liouville operators with an operator potential. In the case when the deficiency indices take indeterminate values, a description of extensions of differential operators was given in [1, 28–30]. The readers may find some papers related to extension theory in [20, 24, 35].

Matrix-valued Sturm–Liouville equations arise in a variety of physical problems (for example, see [5, 10–15, 18, 34]). While there are several results

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about matrix-valued Sturm–Liouville equations, to the best knowledge of the authors of this paper, there is no study on matrix-valued Hahn–Sturm–Liouville operators in the literature. So, in this study, we discuss matrix-valued Hahn–Sturm–Liouville operators. In the analysis that follows, we will largely follow the development of the theory in [1, 5, 8, 21, 25, 27, 32, 35]

This paper is organized as follows. In Section 2, an existence and uniqueness theorem is given. Later, the corresponding maximal and minimal operators for the matrix-valued Hahn–Sturm–Liouville equation are constructed and some properties of these operators are investigated. In Section 3, maximal dissipative, maximal accumulative and self-adjoint extensions of the minimal operator are studied.

Now, we recall some necessary concepts of the Hahn calculus. For more details, the reader may want to consult [2–4, 6, 7, 22, 23].

Throughout the paper, we let $\omega > 0$ and $q \in (0,1)$. Let I be a real interval containing ω_0, where $\omega_0 := \frac{\omega}{1-q}$.

Definition 1 ([22, 23]). Let $u : I \to \mathbb{R}$ be a function. If u is differentiable at ω_0, then the Hahn difference operator $D_{\omega,q}$ is given by the formula

$$D_{\omega,q}u(x) = \begin{cases} [\omega + (q-1)x]^{-1}[u(\omega + qx) - u(x)], & x \neq \omega_0, \\ u'(\omega_0), & x = \omega_0. \end{cases}$$

Theorem 2 ([6]). *Let $u, v : I \to \mathbb{R}$ be ω, q-differentiable at $x \in I$. Then we have*

i) $D_{\omega,q}(au + bv)(x) = aD_{\omega,q}u(x) + bD_{\omega,q}v(x), \ a, b \in I,$

ii) $D_{\omega,q}(u/v)(x) = \dfrac{D_{\omega,q}(u(x))v(x) - u(x)D_{\omega,q}v(x)}{v(x)v(\omega + xq)},$

iii) $D_{\omega,q}(uv)(x) = (D_{\omega,q}u(x))v(x) + u(\omega + xq)D_{\omega,q}v(x),$

iv) $D_{\omega,q}u\left(h^{-1}(x)\right) = D_{-\omega q^{-1}, q^{-1}}u(x),$
where $h(x) := \omega + qx$, $h^{-1}(x) = q^{-1}(x - \omega)$, *and* $x \in I$.

Definition 3 ([6]). Let $u : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. The ω, q-integral of u is given by

$$\int_a^b u(x)d_{\omega,q}x := \int_{\omega_0}^b u(x)d_{\omega,q}x - \int_{\omega_0}^a u(x)d_{\omega,q}x,$$

where

$$\int_{\omega_0}^x u(x)d_{\omega,q}x := ((1-q)x - \omega)\sum_{n=0}^{\infty} q^n u\left(\omega\frac{1-q^n}{1-q} + xq^n\right), \ x \in I$$

provided that the series converges.

2. The matrix-valued Hahn–Sturm–Liouville problem. Consider the matrix-valued Hahn–Sturm–Liouville equation defined as

$$\begin{aligned} l_1(z) &:= -\frac{1}{q} D_{-\omega q^{-1}, q^{-1}} \left[P(x) D_{\omega,q} z(x) \right] + Q_1(x) z(x) \\ &= \lambda V_1(x) z(x), \; x \in [\omega_0, a], \end{aligned} \tag{1}$$

where P, V_1 and Q_1 are $n \times n$ complex Hermitian matrix-valued functions defined and continuous on $[\omega_0, h^{-1}(a)]$, $\det P(x) \neq 0$, $P^{-1}(x)$ is continuous on $[\omega_0, h^{-1}(a)]$, $V_1(x)$ is positive, and λ is a complex parameter.

Let

$$\begin{aligned} \mathcal{Z}(x) &= \begin{pmatrix} z(x) \\ P(x) D_{\omega,q} z(x) \end{pmatrix}, \\ \mathcal{Z}^{[h]}(x) &= \begin{pmatrix} D_{\omega,q} z(x) \\ \frac{1}{q} D_{-\omega q^{-1}, q^{-1}} \left(P(x) D_{\omega,q} z(x) \right) \end{pmatrix}, \\ V(x) &= \begin{pmatrix} V_1(x) & O_n \\ O_n & O_n \end{pmatrix}, \quad W(x) = \begin{pmatrix} -Q_1(x) & O_n \\ O_n & P^{-1}(x) \end{pmatrix}. \end{aligned}$$

Then we can transform Eq. (1) into the following Hahn–Hamiltonian system

$$l(\mathcal{Z}) := J \mathcal{Z}^{[h]}(x) - W(x) \mathcal{Z}(x) = \lambda V(x) \mathcal{Z}(x), \; x \in [\omega_0, a], \tag{2}$$

where

$$J = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix},$$

and I_n (O_n) is the identity (zero) matrix on $\mathbb{C}^n$.

Let

$$L^2_{\omega,q,V}\left[(\omega_0, a); E\right] = \left\{ \mathcal{Z} : \int_{\omega_0}^{a} (V\mathcal{Z}, \mathcal{Z})_E d_{\omega,q} x < \infty \right\}$$

be the Hilbert space of vector-valued functions $\mathcal{Z}, \mathcal{Y}$ with the inner product

$$\begin{aligned} (\mathcal{Z}, \mathcal{Y}) &:= \int_{\omega_0}^{a} (V\mathcal{Z}, \mathcal{Y})_E d_{\omega,q} x \\ &= \int_{\omega_0}^{a} \mathcal{Y}^*(x) V(x) \mathcal{Z}(x) d_{\omega,q} x, \end{aligned}$$

where $E := \mathbb{C}^{2n}$ is the $2n$-dimensional Euclidean space, and * indicates the complex conjugate transpose.

Let

$$C^2_{\omega,q}\left[(\omega_0, a); E\right] = \left\{ \mathcal{Z} : z \text{ and } P(x) D_{\omega,q} z \text{ are continuous at } \omega_0 \right\}.$$

It is evident that $C^2_{\omega,q}\left[(\omega_0, a); E\right] \subset L^2_{\omega,q,V}\left[(\omega_0, a); E\right]$.

Theorem 4. *For $K \in \mathbb{C}^{2n}$, Eq. (2) with the initial condition*

$$\mathcal{Z}(\omega_0, \lambda) = K \quad (\lambda \in \mathbb{C}) \tag{3}$$

has a unique solution in $C^2_{\omega,q}[(\omega_0, a); E]$.

Proof. From (2), we see that

$$\mathcal{Z}(x,\lambda)=K-q\int_{\omega_0}^{x}J[\lambda V(h(t),\lambda)+W(h(t),\lambda)]\mathcal{Z}(h(t),\lambda)d_{\omega,q}t, \tag{4}$$

where $x\in[\omega_0,a]$.

Define the sequence $\{\mathcal{Z}_i\}_{i=1}^{\infty}$ of successive approximations by

$$\mathcal{Z}_0(x,\lambda)=K,$$

$$\mathcal{Z}_{i+1}(x,\lambda)=K-q\int_{\omega_0}^{x}J[\lambda V(h(t),\lambda)+W(h(t),\lambda)]\mathcal{Z}_i(h(t),\lambda)d_{\omega,q}t, \tag{5}$$

where $i=0,1,2,\dots$ and $x\in[\omega_0,a]$.

Now, we prove that $\{\mathcal{Z}_i\}_{i=1}^{\infty}$ converges uniformly on each compact subset of $[\omega_0,a]$. There exist positive numbers $\varkappa(\lambda)$ and $\varrho(\lambda)$ such that

$$\|J[\lambda V(h(x),\lambda)+W(h(x),\lambda)]\|_E\leq\varkappa(\lambda),$$
$$\|\mathcal{Z}_1(x,\lambda)\|_E\leq\varrho(\lambda),$$

where $x\in[\omega_0,a]$. Using mathematical induction, we get

$$\|\mathcal{Z}_{i+1}(x,\lambda)-\mathcal{Z}_i(x,\lambda)\|_E\leq C\varkappa(\lambda)q^{\frac{i(i+1)}{2}}\frac{(\varrho(\lambda)(x-\omega_0)(1-q))^i}{(q;q)_i},$$

where $i\in\mathbb{N}$, $C>0$ and

$$(q;q)_i=\prod_{k=0}^{i-1}\left(1-q^{k+1}\right).$$

The Weierstrass M-test now shows that the sequence $\{\mathcal{Z}_i\}_{i=1}^{\infty}$ converges to a function $\mathcal{Z}$ uniformly on each compact subset of $[\omega_0,a]$. One can prove that z and $D_{\omega,q}z$ are continuous at ω_0. It is obvious that the function $\mathcal{Z}$ satisfies (3).

We proceed to show that (2) has a unique solution. Suppose that $\mathcal{Y}$ is another solution of (2). Proceeding as above, we conclude that

$$\|\mathcal{Z}(x,\lambda)-\mathcal{Y}(x,\lambda)\|_E\leq C_1\varkappa(\lambda)q^{\frac{i(i+1)}{2}}\frac{(\varrho(\lambda)(x-\omega_0)(1-q))^i}{(q;q)_i},$$

where $i\in\mathbb{N}$, $C_1>0$. Then we obtain

$$\lim_{i\to\infty}q^{\frac{i(i+1)}{2}}\frac{(\varrho(\lambda)(x-\omega_0)(1-q))^i}{(q;q)_i}=0.$$

We thus get $\mathcal{Z}=\mathcal{Y}$ on $[\omega_0,a]$. □

Now, we will introduce the definition of maximal and minimal operators for Eq. (2).

Denote

$$\mathcal{D}_{\max} := \left\{ \mathcal{Z} \in L^2_{\omega,q,V}\left[(\omega_0, a); E\right] : \begin{array}{l} \mathcal{Z} \text{ is continuous at } \omega_0, \\ J\mathcal{Z}^{[h]}(x) - W(x)\mathcal{Z}(x) = VF \\ \text{exists in } [\omega_0, a], \text{ and} \\ F \in L^2_{\omega,q,V}[(\omega_0, a); E] \end{array} \right\}$$

and

$$\mathcal{D}_{\min} := \left\{ \mathcal{Z} \in \mathcal{D}_{\max} : \widehat{\mathcal{Z}}(\omega_0) = \widehat{\mathcal{Z}}(a) = 0 \right\}, \tag{6}$$

where

$$\widehat{\mathcal{Z}}(x) = \begin{pmatrix} z(x) \\ P\left(h^{-1}(x)\right) D_{\omega,q} z\left(h^{-1}(x)\right) \end{pmatrix}.$$

The minimal operator $T_{\min}$ is defined by

$$\begin{aligned} T_{\min} : \mathcal{D}_{\min} &\to L^2_{\omega,q,V}[(\omega_0, a); E], \\ \mathcal{Z} &\to T_{\min}\mathcal{Z} = F \end{aligned}$$

if and only if $l(\mathcal{Z}) = VF$ for all $\mathcal{Z} \in \mathcal{D}_{\min}$. The maximal operator $T_{\max}$ is defined by

$$\begin{aligned} T_{\max} : \mathcal{D}_{\max} &\to L^2_{q,V}[(\omega_0, a); E], \\ \mathcal{Z} &\to T_{\max}\mathcal{Z} = F \end{aligned}$$

if and only if $l(\mathcal{Z}) = VF$ for all for all $\mathcal{Z} \in \mathcal{D}_{\max}$.

The Green formula is given by the next theorem.

Theorem 5. *Let $\mathcal{U}, \mathcal{Y} \in \mathcal{D}_{\max}$. Then we have*

$$\begin{aligned} \int_{\omega_0}^{t} \left[\mathcal{Y}^*(x) J\mathcal{U}^{[h]}(x) - \left\{J\mathcal{Y}^{[h]}(x)\right\}^* \mathcal{U}(x)\right] d_{\omega,q}x \\ = \widehat{\mathcal{Y}}^*(t) J\widehat{\mathcal{U}}(t) - \widehat{\mathcal{Y}}^*(\omega_0) J\widehat{\mathcal{U}}(\omega_0), \end{aligned}$$

where $t \in (\omega_0, a]$.

Proof.

$$\begin{aligned} &\int_{\omega_0}^{t} \left[\mathcal{Y}^*(x) J\mathcal{U}^{[h]}(x) - \left\{J\mathcal{Y}^{[h]}(x)\right\}^* \mathcal{U}(x)\right] d_{\omega,q}x \\ &= \int_{\omega_0}^{t} \left\{ \begin{array}{c} \begin{pmatrix} y(x) \\ P(x)D_{\omega,q}y(x) \end{pmatrix}^* \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix} \\ \times \begin{pmatrix} D_{\omega,q}u(x) \\ \frac{1}{q} D_{-\omega q^{-1}, q^{-1}}\left(P(x)D_{\omega,q}u(x)\right) \end{pmatrix} d_{\omega,q}x \end{array} \right\} \\ &\quad - \int_{\omega_0}^{t} \left\{ \begin{array}{c} \left[\begin{pmatrix} D_{\omega,q}y(x) \\ \frac{1}{q} D_{-\omega q^{-1}, q^{-1}}\left(P(x)D_{\omega,q}y(x)\right) \end{pmatrix}\right]^* \\ \times \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix} \begin{pmatrix} u(x) \\ P(x)D_{\omega,q}u(x) \end{pmatrix} d_{\omega,q}x \end{array} \right\} \end{aligned}$$

$$= \int_{\omega_0}^{t} \left[\begin{array}{c} y^*(x) \left\{ \frac{1}{q} D_{-\omega q^{-1}, q^{-1}} \left(P(x) D_{\omega,q} u(x) \right) \right\} \\ + \left(P(x) D_{\omega,q} y(x) \right)^* D_{\omega,q} u(x) \end{array} \right] d_{\omega,q} x$$
$$- \int_{\omega_0}^{t} \left[\begin{array}{c} \left\{ -\frac{1}{q} D_{-\omega q^{-1}, q^{-1}} \left(P(x) D_{\omega,q} y(x) \right) \right\}^* u(x) \\ + \left(D_{\omega,q} y(x) \right)^* P(x) D_{\omega,q} u(x) \end{array} \right] d_{\omega,q} x$$
$$= \int_{\omega_0}^{t} D_{\omega,q} \left\{ \begin{array}{c} \left[(P D_{\omega,q} y) \left(h^{-1}(x) \right) \right]^* u(x) \\ - y^*(x) \left(P D_{\omega,q} u \right) \left(h^{-1}(x) \right) \end{array} \right\} d_{\omega,q} x$$
$$= \widehat{\mathcal{Y}}^*(t) J \widehat{\mathcal{U}}(t) - \widehat{\mathcal{Y}}^*(\omega_0) J \widehat{\mathcal{U}}(\omega_0).$$

□

Then by Theorem 5, the following theorem is obtained.

Theorem 6. *For all $\mathcal{U}, \mathcal{Y} \in \mathcal{D}_{\max}$ we have*

$$(T_{\max}\mathcal{U}, \mathcal{Y}) - (\mathcal{U}, T_{\max}\mathcal{Y}) = [\mathcal{U}, \mathcal{Y}]_a - [\mathcal{U}, \mathcal{Y}]_{\omega_0}, \tag{7}$$

where $[\mathcal{U}, \mathcal{Y}]_x := \widehat{\mathcal{Y}}^*(x) J \widehat{\mathcal{U}}(x)$, $x \in [\omega_0, a]$.

Lemma 7. *The minimal operator $T_{\min}$ is Hermitian.*

Proof. For $\mathcal{U}, \mathcal{Y} \in \mathcal{D}_{\min}$, there exist $F, G \in L^2_{q,V}[(\omega_0, a) ; E]$ such that $l(\mathcal{U}) = VF$ and $l(\mathcal{Y}) = VG$. It follows from (6) and (7) that

$$\begin{aligned} (T_{\min}\mathcal{U}, \mathcal{Y}) - (\mathcal{U}, T_{\min}\mathcal{Y}) &= (F, \mathcal{Y}) - (\mathcal{U}, G) \\ &= \int_{\omega_0}^{a} \left[\mathcal{Y}^*(t) V F - G^*(t) V \mathcal{U}(t) \right] d_{\omega,q} t \\ &= \int_{\omega_0}^{a} \left[\mathcal{Y}^*(t) l(\mathcal{U}) - l(\mathcal{Y})^* \mathcal{U}(t) \right] d_{\omega,q} t \\ &= [\mathcal{U}, \mathcal{Y}]_a - [\mathcal{U}, \mathcal{Y}]_{\omega_0} = 0. \end{aligned}$$

□

A proof of the following lemma is similar to that of Lemma 7.

Lemma 8. *For all $\mathcal{U} \in \mathcal{D}_{\min}$ and for all $\mathcal{Y} \in \mathcal{D}_{\max}$, we have the relation*

$$(T_{\min}\mathcal{U}, \mathcal{Y}) = (\mathcal{U}, T_{\max}\mathcal{Y}).$$

Lemma 9. *Let $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the null space and the range of an operator T, respectively. Then we have*

$$\mathcal{R}(T_{\min}) = \mathcal{N}(T_{\max})^{\perp}.$$

Proof. Given any $\xi \in \mathcal{R}(T_{\min})$, there exists $\mathcal{U} \in \mathcal{D}_{\min}$ such that $T_{\min}\mathcal{U} = \xi$. From Lemma 8, it follows that

$$(\xi, \mathcal{Y}) = (T_{\min}\mathcal{U}, \mathcal{Y}) = (\mathcal{U}, T_{\max}\mathcal{Y}) = 0,$$

for each $\mathcal{Y} \in \mathcal{N}(T_{\max})$. Thus $\xi \in \mathcal{N}(T_{\max})^{\perp}$. For any given $\xi \in \mathcal{N}(T_{\max})^{\perp}$ and for all $\mathcal{Y} \in \mathcal{N}(T_{\max})$, we have $(\xi, \mathcal{Y}) = 0$.

Consider the problem:

$$J\mathcal{Z}^{[h]}(x) - W(x)\mathcal{Z}(x) = V(x)\xi(x), \ \widehat{\mathcal{Z}}(\omega_0) = 0, \tag{8}$$

where $x \in [\omega_0, a]$. According to Theorem 4, (8) has a unique solution on $[\omega_0, a]$. Let $\Psi(x) = (\psi_1, \psi_2, \ldots, \psi_{2n})$ be the fundamental solution of the system

$$J\mathcal{Z}^{[h]}(x) - W(x)\mathcal{Z}(x) = 0, \ \widehat{\Psi}(a) = J,$$

where $x \in [\omega_0, a]$. It follows easily that $\psi_k \in \mathcal{N}(T_{\max})$ for $1 \le k \le 2n$. Therefore for $1 \le k \le 2n$,

$$\begin{aligned} 0 = (\xi, \psi_k) &= \int_{\omega_0}^{a} \psi_k^*(t)V(x)\xi(t)d_{\omega,q}t = \int_{\omega_0}^{a} \psi_k^*(t)l(\mathcal{Z})(t)d_{\omega,q}t \\ &= \int_{\omega_0}^{a} \psi_k^*(t)l(\mathcal{Z})(t)d_{\omega,q}t - \int_{\omega_0}^{a} l\left(\psi_k\right)^*(t)\mathcal{Z}(t)d_{\omega,q}t \\ &= [\mathcal{Z}, \psi_k]_a - [\mathcal{Z}, \psi_k]_{\omega_0} = [\mathcal{Z}, \psi_k]_a = \widehat{\psi_k}^*(a)J\widehat{\mathcal{Z}}(a), \end{aligned}$$

by Theorem 6. This gives $\widehat{\Psi}^*(a)J\widehat{\mathcal{Z}}(a) = \widehat{\mathcal{Z}}(a) = 0$, i.e., $\xi \in \mathcal{R}(T_{\min})$. □

Theorem 10. *The operator $T_{\min}$ is a densely defined and symmetric operator. Furthermore $T^*_{\min} = T_{\max}$, where $T^*_{\min}$ denotes the adjoint operator of $T_{\min}$.*

Proof. Assume that $\xi \in \mathcal{D}^{\perp}_{\min}$. Then, for all $\mathcal{Y} \in \mathcal{D}_{\min}$, we have $(\xi, \mathcal{Y}) = 0$. Write $T_{\min}\mathcal{Y}(x) = \phi(x)$. We will denote by $\mathcal{U}(x)$ any solution of the Hahn–Hamiltonian system

$$J\mathcal{U}^{[h]}(x) - W(x)\mathcal{U}(x) = V(x)\xi(x),$$

where $x \in [\omega_0, a]$. Theorem 6 now yields

$$\begin{aligned} &(\mathcal{U}, \phi) - (\xi, \mathcal{Y}) \\ &\quad = \int_{\omega_0}^{a} \phi^*(t)V(t)\mathcal{U}(t)d_{\omega,q}t - \int_{\omega_0}^{a} \mathcal{Y}^*(t)V(t)\xi(t)d_{\omega,q}t \\ &\quad = \int_{\omega_0}^{a} l(\mathcal{Y})^*(t)\mathcal{U}(t)d_{\omega,q}t - \int_{\omega_0}^{a} \mathcal{Y}^*(t)l(\mathcal{U})(t)d_{\omega,q}t \\ &\quad = -[\mathcal{U}, \mathcal{Y}]_a + [\mathcal{U}, \mathcal{Y}]_{\omega_0} = 0. \end{aligned}$$

Therefore $\mathcal{U} \in \mathcal{R}(T_{\min})^{\perp} = \mathcal{N}(T_{\max})$ by Lemma 9. From this, it follows that $\xi = 0$, i.e., $\mathcal{D}^{\perp}_{\min} = \{0\}$. According to Lemma 7, $T_{\min}$ is a symmetric operator.

Let $\mathcal{D}^*_{\min}$ be the domain of $T^*_{\min}$. Our next goal is to we prove that $\mathcal{D}^*_{\min} = \mathcal{D}_{\max}$, and $T^*_{\min}\mathcal{U} = T_{\max}\mathcal{U}$ for all $\mathcal{U} \in \mathcal{D}^*_{\min}$. From Lemma 8, for any given $\mathcal{U} \in \mathcal{D}_{\max}$, we have $(\mathcal{U}, T_{\min}\mathcal{Y}) = (T_{\max}\mathcal{U}, \mathcal{Y}) = (T^*_{\min}, \mathcal{Y})$ for all $\mathcal{Y} \in \mathcal{D}_{\min}$. Then $\mathcal{U} \in \mathcal{D}^*_{\min}$, i.e., $\mathcal{D}_{\max} \subset \mathcal{D}^*_{\min}$.

Let $\mathcal{U} \in \mathcal{D}_{\min}^{*}$. Then we have $\mathcal{U}, \phi \in L_{q,V}^{2}\left[(\omega_0, a) ; E\right]$, where $\phi := T_{\min}^{*}\mathcal{U}$. Suppose that Φ is a solution of the system

$$J\Phi^{[h]}(x) - W(x)\Phi(x) = V(x)\phi(x). \tag{9}$$

By Lemma 8, it follows that

$$(\phi, \mathcal{Y}) = (T_{\max}\Phi, \mathcal{Y}) = (\Phi, T_{\min}\mathcal{Y}).$$

Hence

$$\begin{aligned}(\mathcal{U} - \Phi, T_{\min}\mathcal{Y}) &= (\mathcal{U}, T_{\min}\mathcal{Y}) - (\Phi, T_{\min}\mathcal{Y}) \\ &= (T_{\min}^{*}\mathcal{U}, \mathcal{Y}) - (\phi, \mathcal{Y}) = 0,\end{aligned}$$

i.e., $\mathcal{U} - \Phi \in \mathcal{R}\left(T_{\min}\right)^{\perp}$. From Lemma 9, we obtain $\mathcal{U} - \Phi \in \mathcal{N}\left(T_{\max}\right)$.

From (9), it may be concluded that

$$\begin{aligned}J\mathcal{U}^{[h]}(x) - W(x)\mathcal{U}(x) &= J\Phi^{[q]}(x) - W(x)\Phi(x) \\ &= V(x)\phi(x),\end{aligned}$$

where $x \in [\omega_0, a]$. As $\mathcal{U}, \phi \in L_{q,V}^{2}[(\omega_0, a); E]$, we see that $\mathcal{U} \in \mathcal{D}_{\max}$ and $T_{\max}\mathcal{U} = \phi = T_{\min}^{*}\mathcal{U}$. This completes the proof. $\square$

3. Extensions of the symmetric operator. In this section, we introduce the maximal dissipative, maximal accumulative and self-adjoint extensions of the symmetric operator $T_{\min}$.

We begin this section with the following definition (see [16, 21, 24]).

Definition 11. Let $\mathbb{H}$ be a Hilbert space; let Π_1 and Π_2 be linear mappings of $\mathcal{D}\left(\mathcal{B}^{*}\right)$ into $\mathbb{H}$, where $\mathcal{B}$ is a closed symmetric operator acting in a Hilbert space $\mathcal{H}$ with equal (finite or infinite) deficiency indices. Then the triplet $(\mathbb{H}, \Pi_1, \Pi_2)$ is called a *space of boundary values* of the operator $\mathcal{B}$ if

1. $(\mathcal{B}^{*}h, g)_{\mathcal{H}} - (h, \mathcal{B}^{*}g)_{\mathcal{H}} = (\Pi_1 h, \Pi_2 g)_{\mathbb{H}} - (\Pi_2 h, \Pi_1 g)_{\mathbb{H}}, \forall h, g \in \mathcal{D}((\mathcal{B}^{*})$, and
2. for every $G_1, G_2 \in \mathbb{H}$, there exists a vector $g \in \mathcal{D}(\mathcal{B}^{*})$ such that $\Pi_1 g = G_1$ and $\Pi_2 g = G_2$.

Let

$$\Pi_1, \Pi_2 : \mathcal{D}_{\max} \to \mathbb{C}^n \oplus \mathbb{C}^n,$$

where

$$\Pi_1\mathcal{U} = \begin{pmatrix} -u(\omega_0) \\ u(a) \end{pmatrix}, \ \Pi_2\mathcal{U} = \begin{pmatrix} (PD_{-\omega q^{-1}, q^{-1}}u)(\omega_0) \\ P(h^{-1}(a))D_{-\omega q^{-1}, q^{-1}}u(a) \end{pmatrix}, \tag{10}$$

and

$$\mathcal{U}(x) = \begin{pmatrix} u(x) \\ P(x)D_{-\omega q^{-1}, q^{-1}}u(h(x)) \end{pmatrix},$$

$\mathcal{U} \in \mathcal{D}_{\max}$.

Theorem 12. *The triplet* $(\mathbb{C}^n \oplus \mathbb{C}^n, \Pi_1, \Pi_2)$ *defined by* (10) *is a space of boundary values of the symmetric operator* $T_{\min}$.

Proof. From (10) and (7), we see that

$$\begin{aligned}
&(\Pi_1\mathcal{U},\Pi_2\mathcal{Y})_{\mathbb{C}^n\oplus\mathbb{C}^n}-(\Pi_2\mathcal{U},\Pi_1\mathcal{Y})_{\mathbb{C}^n\oplus\mathbb{C}^n}\\
&=-\big(u(\omega_0),(PD_{-\omega q^{-1},q^{-1}}y)(\omega_0)\big)_{\mathbb{C}^n}+\big(u(a),P(h^{-1}(a))D_{-\omega q^{-1},q^{-1}}y(a)\big)_{\mathbb{C}^n}\\
&\quad+\big((PD_{-\omega q^{-1},q^{-1}}u)(\omega_0),y(\omega_0)\big)_{\mathbb{C}^n}-\big(P(h^{-1}(a))D_{-\omega q^{-1},q^{-1}}u(a),y(a)\big)_{\mathbb{C}^n}\\
&=[\mathcal{U},\mathcal{Y}](a)-[\mathcal{U},\mathcal{Y}](\omega_0)=(T_{\max}\mathcal{U},\mathcal{Y})-(\mathcal{U},T_{\max}\mathcal{Y}),
\end{aligned}$$

where $\mathcal{U},\mathcal{Y}\in\mathcal{D}_{\max}$.

Our next goal is to show the second assumption of the definition of space of boundary values. Let

$$\Lambda=\begin{pmatrix}\Lambda_1\\ \Lambda_2\end{pmatrix},\ \Gamma=\begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix}\in\mathbb{C}^n\oplus\mathbb{C}^n.$$

We construct the vector-valued function

$$u(x)=\alpha_1(x)\circ\Lambda_1+\alpha_2(x)\circ\Gamma_1+\beta_1(x)\circ\Lambda_2+\beta_2(x)\circ\Gamma_2,$$

where $\circ$ is a symbol of the Hadamard product of vectors and the vector-valued functions $\alpha_i(x)$, $\beta_i(x)$, $D_{-\omega q^{-1},q^{-1}}\alpha_i(x)$, $D_{-\omega q^{-1},q^{-1}}\beta_i(x)\in\mathbb{R}^n$ ($i=1,2$) are defined on $[\omega_0,h^{-1}(a)]$ and continuous at ω_0 which satisfies the conditions:

$$\begin{aligned}
&\alpha_{1j}(\omega_0)=-1,\ \alpha_1(a)=0,\ D_{-\omega q^{-1},q^{-1}}\alpha_1(\omega_0)=0,\ D_{-\omega q^{-1},q^{-1}}\alpha_1(a)=0,\\
&\alpha_2(\omega_0)=0,\ \alpha_2(a)=0,\ (PD_{-\omega q^{-1},\,q^{-1}}\alpha_2)_j(\omega_0)=1,\ D_{-\omega q^{-1},q^{-1}}\alpha_2(a)=0,\\
&\beta_1(\omega_0)=0,\ \beta_{1j}(a)=1,\ D_{-\omega q^{-1},q^{-1}}\beta_1(\omega_0)=0,\ D_{-\omega q^{-1},q^{-1}}\beta_1(a)=0,\\
&\beta_2(\omega_0)=0,\ \beta_2(a)=0,\ D_{-\omega q^{-1},\,q^{-1}}\beta_2(\omega_0)=0,\\
&P(h^{-1}(a))(D_{-\omega q^{-1},q^{-1}}\beta_2)_j(a)=1\quad(j=1,2,\dots,n).
\end{aligned}$$

Then we have

$$\mathcal{U}(x)=\begin{pmatrix}u(x)\\ P(x)D_{-\omega q^{-1},q^{-1}}u(h(x))\end{pmatrix}=\begin{pmatrix}u(x)\\ P(x)D_{\omega,q}u(x)\end{pmatrix},$$

$\mathcal{U}\in\mathcal{D}_{\max}$ and $\Pi_1\mathcal{U}=\Lambda,\ \Pi_2\mathcal{U}=\Gamma$. This completes the proof. □

Now, we give the following definition.

Definition 13 ([21]). Let $\mathcal{L}$ be a linear operator with dense domain $\mathcal{D}(\mathcal{L})$ acting on some Hilbert space $\mathcal{H}$. The operator $\mathcal{L}$ is called *dissipative* if

$$\operatorname{Im}(\mathcal{L}f,f)\ge 0$$

for all $f\in\mathcal{D}(\mathcal{L})$ and is called *maximal dissipative* if it does not have a proper dissipative extension. Similarly, the operator $\mathcal{L}$ is called *accumulative* if

$$\operatorname{Im}(\mathcal{L}f,f)\le 0$$

for all $f\in\mathcal{D}(\mathcal{L})$ and is called *maximal accumulative* if it does not have a proper accumulative extension.

Let

$$D_1 = \{\mathcal{Y} \in \mathcal{D}_{\max} : (M - I)\Pi_1\mathcal{Y} + i(M + I)\Pi_2\mathcal{Y} = 0\}, \tag{11}$$

$$D_2 = \{\mathcal{Y} \in \mathcal{D}_{\max} : (M - I)\Pi_1\mathcal{Y} - i(M + I)\Pi_2\mathcal{Y} = 0\}, \tag{12}$$

where M is a contraction operator in $\mathbb{C}^n \oplus \mathbb{C}^n$.

Then by Theorem 12, the following theorem is obtained [21].

Theorem 14. *The restriction of the maximal operator $T_{\max}$ to the set D_1 is a maximal dissipative extension of the symmetric operator $T_{\min}$. Conversely, any maximal dissipative extensions of $T_{\min}$ is the restriction of $T_{\max}$ to a set D_1. Similarly, the restriction of the operator $T_{\max}$ to the set D_2 is a maximal accumulative extension of the symmetric operator $T_{\min}$. Conversely, any maximal accumulative extensions of $T_{\min}$ is the restriction of $T_{\max}$ to a set D_2. Here, the contraction M is uniquely determined by the extension. If the operator M is unitary, these conditions define a self-adjoint extension of $T_{\min}$.*

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