doi: 10.17951/a.2024.78.1.37-73

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXXVIII, NO. 1, 2024 SECTIO A 37–73

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Applications of quadratic and cubic hypergeometric transformations

Abstract. The purpose of this paper is to consider five classes of quadratic and cubic hypergeometric transformations in the spirit of Bailey and Whipple. We shall successfully evaluate several hypergeometric functions, of the types ${}_2\mathrm{F}_1(x)$, ${}_3\mathrm{F}_2(x)$, and ${}_4\mathrm{F}_3(x)$, with each function having one or more free parameters, and with the argument x chosen to be equal to such unusual values as $x = \pm 1, -8, \frac{1}{4}, -\frac{1}{8}$, (these four values having been explored initially by Gessel and Stanton). In each case, companion identities and/or inverse transformations are given, which are sometimes proved by a limiting process for a divergent hypergeometric series. Some of the proofs use the Clausen quadratic formula, Euler reflection formula, Legendre duplication, Gauss multiplication formula, Euler transformation, hypergeometric reversion formula and known hypergeometric summation formulas. The proofs in the terminating case are simpler and can lead to mixed summation formulas, which depend on values of a negative integer. Some of the formulas use the Digamma function and a dimension formula is referred to.

2010 *Mathematics Subject Classification.* Primary 33C20; Secondary 33C05, 15A04.

Key words and phrases. Quadratic and cubic hypergeometric transformations, Clausen's quadratic formula, divergent hypergeometric series, L'Hôpital's rule, dimension formula.

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1. Introduction

In a previous joint paper [8], we found hypergeometric formulas for

$$x\in\left\{\pm 1,9,-8,\frac{1}{4},-\frac{1}{3},-\frac{1}{8},\frac{1}{9},\frac{8}{9}\right\}.$$

The present paper is also based on notes of the late Per Karlsson.

The article relies heavily on long-known results of two kinds, namely (1) and (2).

(1) Transformation formulas for the functions ${}_3\mathrm{F}_2(x)$ and ${}_4\mathrm{F}_3(x)$. The transformation theory of ${}_2\mathrm{F}_1(x)$ is classical, and many formulas expressing ${}_2\mathrm{F}_1(R(x))$ in terms of ${}_2\mathrm{F}_1(x)$, where R is a rational function

and the parameters of the two ${}_2\mathrm{F}_1$'s are linearly constrained and related, were obtained in the 19th century by Goursat. The quadratic transformations, for which $\deg R = 2$, were more recently documented in the monograph "Special Functions" of Andrews–Askey–Roy (1999) [1, p. 176 f.], see also Erdélyi, A. [4]. Most importantly, it is known that a very few (exactly three!) of the many $\deg R = 2, 3$ transformations of ${}_2\mathrm{F}_1$ can be extended to ${}_3\mathrm{F}_2$, and there is a "companion" version of each of the two $\deg R = 3$ ones that involves ${}_4\mathrm{F}_3$. The three known transformations of ${}_3\mathrm{F}_2$ are traditionally called "Whipple's quadratic transformation" and "Bailey's first and second cubic transformations." See respectively eqs. (WQ1), (9) and (26).

(2) Evaluation formulas for ${}_3\mathrm{F}_2(1)$, applying when the five parameters of ${}_3\mathrm{F}_2$ are suitably constrained. The parameter space is $\mathbb{C}^5$ and there are several affine subspaces of this parameter space, each of which is of dimension 3, on which a "gamma quotient" evaluation of ${}_3\mathrm{F}_2(1)$ is possible. These evaluations are traditionally called Dixon's, Watson's and Whipple's summation formulas, see Appendix.

The paper is divided into five sections, which all have a similar structure. All definitions used in the paper can be found in Appendix. In the first three Sections 2–4, known transformations are used to find variants of Whipple's formulas, Section 2, and new transformations and summations, which are proved by complex computations, the Nørlund limit formula (18), as well as L'Hôpital's rule in Sections 3, 4 are used in the proofs. We shall find "companions" of Bailey's cubic transformations, which relate certain doubly parametrized functions of the type ${}_3\mathrm{F}_2(R(x))$ to corresponding functions of the type ${}_4\mathrm{F}_3(x)$ (see, e.g., eq. (10)). Each such relationship is handled by expressing the ${}_4\mathrm{F}_3$ as a sum of two ${}_3\mathrm{F}_2$'s, with the aid of "contiguity" or "contiguous function" relations (see eqs. (55), (56)). The resulting ${}_3\mathrm{F}_2(x)$ evaluations (see, for instance, (21)) are not of the simple gamma-quotient form, but they are certainly expressed in terms of the gamma function. Bailey's second cubic transformation is based on the degree-3 rational function $R(x) = R_2(x) = \frac{27x^2}{(4-x)^3}$. Perhaps the most interesting feature of the paper is the derivation and application in Sections 5 and 6 of additional, non-classical transformations of the function ${}_3\mathrm{F}_2$, which are called a "third" and a "fourth" cubic transformation, each having a companion transformation that involves ${}_4\mathrm{F}_3$. We observe that the two formulas (29) and (47) are identical. The "third" cubic transformation is obtained in Section 5 by combining both of Bailey's cubic transformations by solving the algebraic equation $R_1(x_1) = R_2(x_2)$, which appears as eq. (41). In this manner, the transformation of ${}_3\mathrm{F}_2$ appearing as formula (40) is derived. As with formulas (9) and (26), there is a companion transformation to formula (40) that involves the function ${}_4\mathrm{F}_3$, which may be replaced by the combination of a pair of functions of the type ${}_3\mathrm{F}_2$.

2. Whipple's quadratic transformation

We start with Whipple's quadratic transformation [14, 7.1], [11, p. 88]:

$$\text{(WQ1)}\quad \begin{aligned} &{}_3\mathrm{F}_2\left[\begin{array}{c}\alpha,\beta,\gamma\\ 1+\alpha-\beta,1+\alpha-\gamma\end{array}\middle|\, x\right]\\ &= (1-x)^{-\alpha}\ {}_3\mathrm{F}_2\left[\begin{array}{c}\frac{1}{2}\alpha,\frac{1}{2}\alpha+\frac{1}{2},1+\alpha-\beta-\gamma\\ 1+\alpha-\beta,1+\alpha-\gamma\end{array}\middle|\,\frac{-4x}{(1-x)^2}\right],\end{aligned}$$

$$|x|<1,\ \left|\frac{4x}{(1-x)^2}\right|<1.$$

The right hand side has $s=\frac{1}{2}$. Consider the function ${}_3\mathrm{F}_2$ for which the parameter space is $\mathbb{C}^5$. Whipple's quadratic transformation has three free parameters and each of Bailey's cubic transformations has two. So, in each of these transformations the parameter vector of ${}_3\mathrm{F}_2(x)$ and the parameter vector of the corresponding ${}_3\mathrm{F}_2(R(x))$ are constrained to lie in an affine subspace of $\mathbb{C}^5$ of respective dimensionality 3, 2, 2.

If x is such that $R(x)=1$, one can exploit the fact mentioned above: there are three 3-dimensional subspaces of the parameter space $\mathbb{C}^5$ on which a gamma-quotient evaluation of ${}_3\mathrm{F}_2(1)$ is possible. Generically in $\mathbb{C}^5$, the intersection of a 3-dimensional subspace with another 3-dimensional subspace is a 1-dimensional subspace according to the formula

$$\mathrm{Dim}(U+V)=\mathrm{Dim}(U)+\mathrm{Dim}(V)-\mathrm{Dim}(U\cap V).$$

For instance, U could be the subspace of the parameter space $\mathbb{C}^5$ which appears in Whipple's quadratic transformation, eq. (WQ1). Since (WQ1) has three parameters it follows that $\mathrm{Dim}(U)=3$. Suppose that V is the "Whipple subspace" (for which $\mathrm{Dim}(V)=3$; see eq. (53), which also has three free parameters). Then, provided that $\mathrm{Dim}(U+V)=5$, we will have $\mathrm{Dim}(U\cap V)=1$. Under these conditions, an evaluation of ${}_3\mathrm{F}_2(-1)$ with one free parameter may be deduced as eq. (4). In the same way, using the "Watson subspace", one deduces the evaluation (5); and by using a "Dixon subspace", formulas (6), (7), (8) can be deduced. The surprising thing is that by starting with either of Bailey's cubic transformations, for which $\mathrm{Dim}(U)=2$ (there are only two free parameters), one can nonetheless obtain an evaluation of ${}_3\mathrm{F}_2(x)$ (at $x=-\frac{1}{8}$ resp. $x=-8$) with one free parameter, rather than none. For instance, combining the first cubic transformation with the Watson subspace yields (16), and combining with the Whipple subspace yields (17). So, in any of the cubic cases, $\mathrm{Dim}(U)=2$ and $\mathrm{Dim}(V)=3$, but $\mathrm{Dim}(U\cap V)$ equals 1. This is only possible if $\mathrm{Dim}(U+V)=4$ rather than 5, which is not a generic behavior. (For generic subspaces $U,V\subseteq\mathbb{C}^5$ of these dimensions, one would expect $\mathrm{Dim}(U+V)=5$.) This is surprising. Clearly, U and V are not positioned relative to each other, in a generic way.

If we let

$$\alpha \mapsto 2\alpha - 1,\ \beta \mapsto \alpha + \frac{1}{2},\ \gamma \mapsto \alpha - \beta - \frac{1}{2},$$

the function ${}_3\mathrm{F}_2$ on the right-hand side becomes ${}_2\mathrm{F}_1$ like in Bailey [2, (4.08)].

$$(1-x)^{2\alpha-1}\ {}_3\mathrm{F}_2\left[\begin{matrix} 2\alpha-1, \alpha+\frac{1}{2}, \alpha-\beta-\frac{1}{2} \\ \alpha-\frac{1}{2}, \alpha+\beta+\frac{1}{2} \end{matrix} \,\middle|\, x\right] = {}_2\mathrm{F}_1\left[\begin{matrix} \alpha, \beta \\ \alpha+\beta+\frac{1}{2} \end{matrix} \,\middle|\, \frac{-4x}{(1-x)^2}\right].$$

The function ${}_2\mathrm{F}_1$ is symmetric in α and β, and so is the left-hand side, that is, we obtain Bailey's formula

$$\begin{aligned}(1-x)^{2\alpha-1}\ {}_3\mathrm{F}_2&\left[\begin{matrix} 2\alpha-1, \alpha+\frac{1}{2}, \alpha-\beta-\frac{1}{2} \\ \alpha-\frac{1}{2}, \alpha+\beta+\frac{1}{2} \end{matrix} \,\middle|\, x\right] \\ = (1-x)^{2\beta-1}\ {}_3\mathrm{F}_2&\left[\begin{matrix} 2\beta-1, \beta+\frac{1}{2}, \beta-\alpha-\frac{1}{2} \\ \beta-\frac{1}{2}, \alpha+\beta+\frac{1}{2} \end{matrix} \,\middle|\, x\right].\end{aligned} \tag{1}$$

The formula is valid for x in the cut plane [10, §7.41 (16)]. This is seen as a linear transformation of a special ${}_3\mathrm{F}_2$ function, although it is based upon a quadratic transformation. Using $x = -1$ in (1) leads to

$${}_3\mathrm{F}_2\left[\begin{matrix} 2\alpha-1, \alpha+\frac{1}{2}, \alpha-\beta-\frac{1}{2} \\ \alpha-\frac{1}{2}, \alpha+\beta+\frac{1}{2} \end{matrix} \,\middle|\, -1\right] = 2^{1-2\alpha}\Gamma\left[\begin{matrix} \alpha+\beta+\frac{1}{2}, \frac{1}{2} \\ \alpha+\frac{1}{2}, \beta+\frac{1}{2} \end{matrix}\right].$$

That is [12, §III 21].

$${}_3\mathrm{F}_2\left[\begin{matrix} a, 1+\frac{1}{2}a, b \\ \frac{1}{2}a, 1+a-b \end{matrix} \,\middle|\, -1\right] = \Gamma\left[\begin{matrix} 1+a-b, \frac{1}{2}(1+a) \\ 1+a, \frac{1}{2}(1+a)-b \end{matrix}\right].$$

Theorem 2.1. *The inverse* $z = \frac{-4x}{(1-x)^2}$ *for Whipple's transformation* (WQ1) *is given by*

$$\begin{aligned}&{}_3\mathrm{F}_2\left[\begin{matrix} a, a+\frac{1}{2}, b+c-2a-1 \\ b, c \end{matrix} \,\middle|\, z\right] \\ &= \left(\frac{1+\sqrt{1-z}}{2}x\right)^{-2a} {}_3\mathrm{F}_2\left[\begin{matrix} 2a, 1+2a-b, , 1+2a-c \\ b, c \end{matrix} \,\middle|\, \frac{-z}{(1+\sqrt{1-z})^2}\right].\end{aligned}$$

Proof. From $z = \frac{-4x}{(1-x)^2}$, we infer

$$x = -\frac{(1 \mp \sqrt{1-z})^2}{z}.$$

The product of the two roots is 1, we shall use the one which has $|x| < 1$, that is the upper minus sign with the usual interpretation of the square root:

$$x = -\frac{(1-\sqrt{1-z})^2}{z} = -\frac{1}{z}\left(\frac{z}{1+\sqrt{1-z}}\right)^2 = \frac{-z}{\left(1+\sqrt{1-z}\right)^2}, \tag{2}$$

$$1 - x = \frac{2}{1+\sqrt{1-z}}.$$

Finally, let

$$\frac{1}{2}\alpha \mapsto a,\ 1+\alpha-\beta \mapsto b,\ 1+\alpha-\gamma \mapsto c$$

to complete the proof. □

We may note that for $z = 1 + k^2$, $k \in \mathbb{R}_+$, $\sqrt{1-z} = ik$, the variable on the RHS of (2) becomes

$$\frac{-1-k^2}{(1+ik)^2} = -\frac{1-ik}{1+ik},$$

with absolute value 1. For the mapping $x \to z$ the following pairs of numbers apply:

x	z
-1	1
$-\frac{1}{3}$	$\frac{3}{4}$
$-\frac{1}{4}$	$\frac{16}{25}$
$-3+2\sqrt{2}$	$\frac{1}{2}$
$-\frac{1}{8}$	$\frac{32}{81}$
$3-2\sqrt{2}$	-1
$\frac{1}{4}$	$-\frac{16}{9}$
$\frac{1}{2}$	-8
$\frac{3}{4}$	-48
1	$-\infty$

TABLE 1. Table for the inverse Whipple transformation

Put $x = -1$ in (WQ1) to obtain [14]:

$$\begin{aligned} &{}_3\mathrm{F}_2\left[\begin{matrix}\alpha,\beta,\gamma\\ 1+\alpha-\beta, 1+\alpha-\gamma\end{matrix}\middle| -1\right] \\ &= 2^{-\alpha}\,{}_3\mathrm{F}_2\left[\begin{matrix}\frac{1}{2}\alpha, \frac{1}{2}(\alpha+1), 1+\alpha-\beta-\gamma\\ 1+\alpha-\beta, 1+\alpha-\gamma\end{matrix}\right]. \end{aligned} \tag{3}$$

We shall now derive summation formulas with argument -1, as corollaries of Whipple's transformation. Some of these formulas have appeared in a different form in Whipple's papers, and they are quoted in each case.

The Pfaff–Saalschütz summation formula is impossible to handle, since $s=\frac{1}{2}$. Instead, according to Whipple's formula we put

$$\alpha=\frac{1}{2},\ 1+2\alpha-\beta-\gamma=2(1+\alpha-\beta-\gamma),\ \beta+\gamma=1.$$

This gives for RHS

$$\frac{1}{\sqrt{2}}\,{}_3F_2\left[\begin{matrix}\frac{1}{4},\frac{3}{4},\frac{1}{2}\\ \frac{3}{2}-\beta,\frac{1}{2}+\beta\end{matrix}\right]=2^{-\frac{1}{2}}\pi\Gamma\left[\begin{matrix}\frac{3}{2}-\beta,\frac{1}{2}+\beta\\ \frac{3}{8}+\frac{1}{2}\beta,\frac{7}{8}-\frac{1}{2}\beta,\frac{5}{8}+\frac{1}{2}\beta,\frac{9}{8}-\frac{1}{2}\beta\end{matrix}\right],$$

which implies [13, (11.3)]:

$$(4)\quad {}_3F_2\left[\begin{matrix}\frac{1}{2},\beta,1-\beta\\ \frac{3}{2}-\beta,\frac{1}{2}+\beta\end{matrix}\middle|-1\right]=2^{-\frac{1}{2}}\Gamma\left[\begin{matrix}\frac{3}{2}-\beta,\frac{1}{2}+\beta,\frac{1}{2},\frac{1}{2}\\ \frac{3}{8}+\frac{1}{2}\beta,\frac{7}{8}-\frac{1}{2}\beta,\frac{5}{8}+\frac{1}{2}\beta,\frac{9}{8}-\frac{1}{2}\beta\end{matrix}\right].$$

According to Watson's formula we may write

$$1+\alpha-\beta=2(1+\alpha-\beta-\gamma),\ 1+\alpha-\gamma=\frac{1}{2}\left(1+\alpha+\frac{1}{2}\right)=\frac{1}{2}\alpha+\frac{3}{4}$$
$$\Rightarrow\alpha=2\gamma-\frac{1}{2},\ \beta=1+\alpha-2\gamma=\frac{1}{2}.$$

This gives

$$2^{\frac{1}{2}-2\gamma}\,{}_3F_2\left[\begin{matrix}\gamma-\frac{1}{4},\gamma+\frac{1}{4},\gamma\\ 2\gamma,\gamma+\frac{1}{2}\end{matrix}\right]=2^{\frac{1}{2}-2\gamma}\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\gamma+\frac{1}{2},\gamma+\frac{1}{2}\\ \frac{3}{8}+\frac{1}{2}\gamma,\frac{5}{8}+\frac{1}{2}\gamma,\frac{5}{8}+\frac{1}{2}\gamma,\frac{3}{8}+\frac{1}{2}\gamma\end{matrix}\right],$$

which implies [13, (11.2)]:

$$(5)\qquad {}_3F_2\left[\begin{matrix}2\gamma-\frac{1}{2},\frac{1}{2},\gamma\\ 2\gamma,\gamma+\frac{1}{2}\end{matrix}\middle|-1\right]=\frac{1}{\sqrt{2}}\left(\Gamma\left[\begin{matrix}\frac{1}{2}\gamma+\frac{1}{4},\frac{1}{2}\gamma+\frac{3}{4}\\ \frac{1}{2}\gamma+\frac{3}{8},\frac{1}{2}\gamma+\frac{5}{8}\end{matrix}\right]\right)^2.$$

According to Dixon's formula we may write

$$1+\frac{1}{2}\alpha=\frac{3}{2}(1+\alpha)-\beta=2(1+\alpha)-\beta-2\gamma,$$
$$\beta=\alpha+\frac{1}{2},\ 2\gamma=\frac{1}{2}(1+\alpha)\ \Rightarrow\alpha=4\gamma-1,\ \beta=4\gamma-\frac{1}{2}.$$

By applying the Legendre duplication and using the Euler mirror formula, the right hand side will be equal to

$$2^{1-4\gamma}\,{}_3F_2\left[\begin{matrix}2\gamma-\frac{1}{2},-\gamma+\frac{1}{2},2\gamma\\ 3\gamma,\frac{1}{2}\end{matrix}\right]=2^{1-4\gamma}\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{4},\gamma+\frac{3}{4},3\gamma\\ \frac{1}{2}+2\gamma,\frac{3}{4}-\gamma,\frac{1}{4}+2\gamma,\gamma\end{matrix}\right]$$
$$=2^{1-4\gamma}2^{-\left(2\gamma-\frac{1}{2}\right)}\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\gamma+\frac{3}{4},3\gamma,\frac{1}{4}\\ \frac{1}{4}+\gamma,\frac{3}{4}+\gamma,\frac{3}{4}-\gamma,\frac{1}{4}+2\gamma,\gamma\end{matrix}\right]$$
$$=2^{\frac{3}{2}-6\gamma}\Gamma\left[\begin{matrix}\frac{1}{4},3\gamma\\ \frac{1}{4}+2\gamma,\gamma\end{matrix}\right]\sin\left(\frac{\pi}{4}+\pi\gamma\right).$$

This implies that the following three theorems can be formulated.

Theorem 2.2.

$$(6)\quad {}_3F_2\left[\begin{matrix}4\gamma-1,4\gamma-\frac{1}{2},\gamma\\3\gamma,\frac{1}{2}\end{matrix}\middle|-1\right]=2^{\frac{3}{2}-6\gamma}\sin\left(\frac{\pi}{4}+\pi\gamma\right)\Gamma\left[\begin{matrix}\frac{1}{4},3\gamma\\\frac{1}{4}+2\gamma,\gamma\end{matrix}\right].$$

Again, put according to Dixon's formula

$$\frac{3}{2}+\frac{1}{2}\alpha=1+\frac{3}{2}\alpha-\beta=2(1+\alpha)-\beta-2\gamma,$$
$$\beta=\alpha-\frac{1}{2},\ 2\gamma=\frac{1}{2}\alpha+1\ \Rightarrow\alpha=4\gamma-2,\ \beta=4\gamma-\frac{5}{2}.$$

Now the right hand side, by Legendre duplication and Euler mirror formula equals

$$2^{2-4\gamma}\,{}_3F_2\left[\begin{matrix}2\gamma-1,2\gamma-\frac{1}{2},\frac{3}{2}-\gamma\\\frac{3}{2},3\gamma-1\end{matrix}\right]=2^{2-4\gamma}\Gamma\left[\begin{matrix}\frac{3}{2},\frac{1}{4},\gamma+\frac{3}{4},3\gamma-1\\\frac{1}{2}+2\gamma,\frac{7}{4}-\gamma,-\frac{3}{4}+2\gamma,\gamma\end{matrix}\right]$$
$$=\frac{1}{2}2^{2-4\gamma}2^{-\left(2\gamma-\frac{1}{2}\right)}\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\gamma+\frac{3}{4},3\gamma-1,\frac{1}{4}\\\frac{1}{4}+\gamma,\frac{3}{4}+\gamma,\frac{7}{4}-\gamma,-\frac{3}{4}+2\gamma,\gamma\end{matrix}\right]$$
$$=2^{\frac{3}{2}-6\gamma}\Gamma\left[\begin{matrix}\frac{1}{4},3\gamma-1\\-\frac{3}{4}+2\gamma,\gamma\end{matrix}\right]\frac{\sin\left(\pi\gamma-\frac{3}{4}\pi\right)}{\gamma-\frac{3}{4}}.$$

This implies:

Theorem 2.3. [13, (11.1)]

$$(7)\quad {}_3F_2\left[\begin{matrix}4\gamma-2,4\gamma-\frac{5}{2},\gamma\\3\gamma-1,\frac{3}{2}\end{matrix}\middle|-1\right]=2^{\frac{3}{2}-6\gamma}\frac{\sin\left(\pi\gamma-\frac{3}{4}\pi\right)}{\gamma-\frac{3}{4}}\Gamma\left[\begin{matrix}\frac{1}{4},3\gamma-1\\-\frac{3}{4}+2\gamma,\gamma\end{matrix}\right].$$

Finally, put according to Dixon's formula

$$2+\alpha-\beta-\gamma=1+\frac{3}{2}\alpha-\beta=\frac{3}{2}(1+\alpha)-\gamma\ \Rightarrow\ \beta=\gamma-\frac{1}{2},\ \alpha=2-2\gamma.$$

Now the right hand side, by the Legendre duplication, equals

$$2^{2\gamma-2}\,{}_3F_2\left[\begin{matrix}1-\gamma,\frac{7}{2}-4\gamma,\frac{3}{2}-\gamma\\\frac{7}{2}-3\gamma,3-3\gamma\end{matrix}\right]$$
$$=2^{2\gamma-2}\Gamma\left[\begin{matrix}\frac{1}{4},\frac{11}{4}-2\gamma,\frac{7}{2}-3\gamma,3-3\gamma\\\frac{9}{2}-4\gamma,\frac{7}{4}-\gamma,\frac{5}{4}-\gamma,2-2\gamma\end{matrix}\right]$$
$$=2^{2\gamma-2}2^{-\left(\frac{1}{2}-4\gamma\right)+6\gamma-5-\left(2\gamma-\frac{3}{2}\right)}\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{4},\frac{11}{4}-2\gamma,6-6\gamma\\\frac{9}{4}-2\gamma,\frac{11}{4}-2\gamma,\frac{5}{2}-2\gamma,2-2\gamma\end{matrix}\right]$$
$$=2^{10\gamma-9}\Gamma\left[\begin{matrix}\frac{1}{4},6-6\gamma\\\frac{9}{4}-2\gamma,4-4\gamma\end{matrix}\right]2^{-(4\gamma-3)}.$$

This implies by putting $\gamma\to1-\gamma$:

Theorem 2.4.

$$(8)\qquad {}_3F_2\left[\begin{matrix}2\gamma,\frac{1}{2}-\gamma,1-\gamma\\3\gamma,\frac{1}{2}+3\gamma\end{matrix}\middle|-1\right]=2^{-6\gamma}\Gamma\left[\begin{matrix}\frac{1}{4},6\gamma\\\frac{1}{4}+2\gamma,4\gamma\end{matrix}\right].$$

3. Bailey's first cubic transformation

3.1. Statements of the transformations. Bailey's cubic transformations are not explicitly given in the main books on special functions [4] or [9]. In the following, we therefore repeat Bailey's first cubic transformation [2, (4.05)]:

$$
\begin{aligned}
&{}_3\mathrm{F}_2\left[\begin{matrix} 3a, b, 1-b \\ \frac{1}{2}(3a+b+1), 1+\frac{1}{2}(3a-b)\end{matrix}\,\middle|\, x\right] = \\
&= (1-4x)^{-3a}\,{}_3\mathrm{F}_2\left[\begin{matrix} a, a+\frac{1}{3}, a+\frac{2}{3} \\ \frac{1}{2}(3a+b+1), 1+\frac{1}{2}(3a-b)\end{matrix}\,\middle|\, \frac{-27x}{(1-4x)^3}\right].
\end{aligned} \tag{9}
$$

Remark 1. We have $s=\frac{1}{2}$. If $x=-\frac{1}{8}$, then the cubic function to the right is equal to one, i.e. $R(x)=1$. For this reason, it is possible to obtain explicit evaluations of ${}_3\mathrm{F}_2$ at $x=-\frac{1}{8}$. If $x=\frac{1}{4}$, then $R(x)=\infty$, which explains why it is possible to obtain evaluations of ${}_3\mathrm{F}_2$ such as formulas (22)–(25). In this and the following section, which are based on Bailey's two cubic transformations of ${}_3\mathrm{F}_2$, Dixon's summation formula is not used, though Watson's and Whipple's are. This is because applying Dixon's formula would constrain the parameter vector (in $\mathbb{C}^5$) to a 0-dimensional subspace, i.e., to a single value, so that there would be no free parameters.

We also have Gessel & Stanton's companion [6, (5.4)]:

$$
\begin{aligned}
&{}_4\mathrm{F}_3\left[\begin{matrix} 3a, 1+a, b, 1-b \\ a, \frac{1}{2}(3a+b+1), 1+\frac{1}{2}(3a-b)\end{matrix}\,\middle|\, x\right] \\
&= (1+8x)(1-4x)^{-1-3a}\,{}_3\mathrm{F}_2\left[\begin{matrix} a+\frac{1}{3}, a+\frac{2}{3}, a+1 \\ \frac{1}{2}(3a+b+1), 1+\frac{1}{2}(3a-b)\end{matrix}\,\middle|\, \frac{-27x}{(1-4x)^3}\right].
\end{aligned} \tag{10}
$$

In this case, we have $s=-\frac{1}{2}$ and look for the inverse of these two transformations, given by

$$
z=\frac{-27x}{(1-4x)^3}, \quad z(1-4x)^3+27x=0. \tag{11}
$$

x is the root of (11), that is equal to zero for $z=0$. The region of validity is: x must not be on the cut from $\frac{1}{4}$ to $+\infty$, $\frac{-27x}{(1-4x)^3}$ must not be on the cut. Now we may write:

$$
x=-\frac{(1-i\tau)^2}{8(1+i\tau)} \text{ implies } 1-4x=1+\frac{1-2i\tau-\tau^2}{2(1+i\tau)}=\frac{3-\tau^2}{2(1+i\tau)}.
$$

$$
\frac{-27x}{(1-4x)^3}=\frac{(1+\tau^2)^2}{(1-\frac{1}{3}\tau^2)^3}\in(1,\infty] \quad \text{for } \tau^2\in(0,3].
$$

For $-\sqrt{3}\le\tau\le\sqrt{3}$ we include both sides of the cut.

In order to express ${}_4\mathrm{F}_3$ as sums of two functions ${}_3\mathrm{F}_2$, we recall the two contiguity relations (55) and (56).

Corollary 3.1.

$$
\begin{aligned}
&-2\,{}_3F_2\left[\begin{matrix}3a,b,1-b\\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|x\right]\\
&\quad+3\,{}_3F_2\left[\begin{matrix}3a+1,b,1-b\\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|x\right]\\
(12)\quad &=(1+8x)(1-4x)^{-1-3a}\,{}_3F_2\left[\begin{matrix}a+\frac{1}{3},a+\frac{2}{3},a+1\\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|\frac{-27x}{(1-4x)^3}\right]\\
&={}_3F_2\left[\begin{matrix}3a,b,1-b\\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|x\right]\\
&\quad+\frac{12bx(1-b)}{(3a+b+1)(3a-b+2)}\,{}_3F_2\left[\begin{matrix}3a+1,1+b,2-b\\ \frac{1}{2}(3a+b+3),2+\frac{1}{2}(3a-b)\end{matrix}\middle|x\right].
\end{aligned}
$$

Proof. This follows by expressing

$$
{}_4F_3\left[\begin{matrix}3a,1+a,b,1-b\\ a,\frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|x\right]
$$

in three ways using (55), (10), and (56). □

Corollary 3.2.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}3a+1,b,1-b\\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|x\right]\\
(13)\quad &=\frac{1}{3}(1+8x)(1-4x)^{-1-3a}\,{}_3F_2\left[\begin{matrix}a+\frac{1}{3},a+\frac{2}{3},a+1\\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|\frac{-27x}{(1-4x)^3}\right]\\
&\quad+\frac{2}{3}(1-4x)^{-3a}\,{}_3F_2\left[\begin{matrix}a,a+\frac{1}{3},a+\frac{2}{3}\\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|\frac{-27x}{(1-4x)^3}\right].
\end{aligned}
$$

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}3a+1,1+b,2-b\\ \frac{1}{2}(3a+b+3),2+\frac{1}{2}(3a-b)\end{matrix}\middle|x\right]\\
&=\frac{(3a+b+1)(3a-b+2)(1-4x)^{-3a}}{12bx(1-b)}\\
(14)\quad &\times\Bigg[\frac{1+8x}{1-4x}\,{}_3F_2\left[\begin{matrix}a+\frac{1}{3},a+\frac{2}{3},a+1\\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|\frac{-27x}{(1-4x)^3}\right]\\
&\quad-{}_3F_2\left[\begin{matrix}a,a+\frac{1}{3},a+\frac{2}{3}\\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\middle|\frac{-27x}{(1-4x)^3}\right]\Bigg].
\end{aligned}
$$

Proof. Equate formulas (12). □

3.2. The case $x=-\frac{1}{8}$. Put $x=-\frac{1}{8}$ in (9) to get

$$(15)\qquad \begin{aligned} &{}_3F_2\left[\begin{matrix} 3a,b,1-b \\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\,\middle|\,-\frac{1}{8}\right] \\ &=\left(\frac{2}{3}\right)^{3a}{}_3F_2\left[\begin{matrix} a,a+\frac{1}{3},a+\frac{2}{3} \\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b)\end{matrix}\right].\end{aligned}$$

Theorem 3.3.

$$(16)\qquad {}_3F_2\left[\begin{matrix} 3b-1,b,1-b \\ b+\frac{1}{2},2b\end{matrix}\,\middle|\,-\frac{1}{8}\right]=2^{3b-3}\left(\Gamma\left[\begin{matrix}\frac{1}{2}b,\frac{1}{2}+b \\ \frac{1}{2},\frac{3}{2}b\end{matrix}\right]\right)^2.$$

Proof. Put $b=\frac{2}{3}-a$ and use Watson's summation formula in (15) to get

$$\begin{aligned} \text{LHS} &\overset{\text{by[5, p. 83 (3.55)]}}{=} \left(\frac{2}{3}\right)^{3a}\Gamma\left[\begin{matrix}\frac{1}{2},a+\frac{5}{6},a+\frac{5}{6},a+\frac{4}{3}-(a+\frac{5}{6}) \\ \frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+\frac{5}{6},\frac{1}{2}a+\frac{5}{6},\frac{1}{2}a+\frac{1}{2}\end{matrix}\right] \\ &=\left(\frac{2}{3}\right)^{3a}\left(\Gamma\left[\begin{matrix}\frac{1}{2},a+\frac{5}{6},\frac{1}{2}a+\frac{1}{6} \\ \frac{1}{2}(a+\frac{1}{3}),\frac{1}{2}(a+1),\frac{1}{2}a+\frac{5}{6}\end{matrix}\right]\right)^2 \\ &\overset{\text{by(52)}}{=}\left(\frac{2}{3}\right)^{3a}\left(\frac{1}{2\pi 3^{\frac{1}{2}-\frac{3a+1}{2}}}\Gamma\left[\begin{matrix}\frac{1}{2},a+\frac{5}{6},\frac{1}{2}a+\frac{1}{6} \\ \frac{3}{2}a+\frac{1}{2}\end{matrix}\right]\right)^2 \\ &=2^{3a-2}\left(\Gamma\left[\begin{matrix}a+\frac{5}{6},\frac{1}{2}a+\frac{1}{6} \\ \frac{1}{2},\frac{3}{2}a+\frac{1}{2}\end{matrix}\right]\right)^2.\end{aligned}$$

Finally, use $b=a+\frac{1}{3}$ to complete the proof. □

Theorem 3.4.

$$(17)\qquad \begin{aligned} &{}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2}+a,\frac{1}{2}-a \\ 1+\frac{1}{2}a,1-\frac{1}{2}a\end{matrix}\,\middle|\,-\frac{1}{8}\right] \\ &=\frac{1}{2\pi\sqrt{2}}\Gamma\left[\begin{matrix}1+\frac{1}{2}a,1-\frac{1}{2}a,\frac{1}{4}(1+a),\frac{1}{4}(1-a) \\ \frac{3}{4}(1+a),\frac{3}{4}(1-a)\end{matrix}\right].\end{aligned}$$

Proof. Insert $a=\frac{1}{6}$ and use Whipple's summation formula in (15) to get

$$\begin{aligned} &{}_3F_2\left[\begin{matrix}\frac{1}{2},b,1-b \\ \frac{1}{2}b+\frac{3}{4},\frac{5}{4}-\frac{1}{2}b\end{matrix}\,\middle|\,-\frac{1}{8}\right]=\sqrt{\frac{2}{3}}\,{}_3F_2\left[\begin{matrix}\frac{1}{6},\frac{1}{2},\frac{5}{6} \\ \frac{1}{2}b+\frac{3}{4},\frac{5}{4}-\frac{1}{2}b\end{matrix}\right] \\ &\overset{\text{by(53)}}{=}\sqrt{\frac{2}{3}}\pi 2^0\Gamma\left[\begin{matrix}\frac{3}{4}+\frac{1}{2}b,\frac{5}{4}-\frac{1}{2}b,\frac{9}{24}-\frac{1}{4}b,\frac{3}{24}+\frac{1}{4}b \\ \frac{17}{24}-\frac{1}{4}b,\frac{11}{24}+\frac{1}{4}b,\frac{25}{24}-\frac{1}{4}b,\frac{19}{24}+\frac{1}{4}b,\frac{9}{24}-\frac{1}{4}b,\frac{3}{24}+\frac{1}{4}b\end{matrix}\right] \\ &\overset{\text{by}2\times(52)}{=}\frac{\sqrt{\frac{2}{3}}\pi}{2\pi 3^{\frac{1}{2}-\frac{9}{8}+\frac{3}{4}b}2\pi 3^{\frac{1}{2}-\frac{3}{8}-\frac{3}{4}b}}\Gamma\left[\begin{matrix}\frac{3}{4}+\frac{1}{2}b,\frac{5}{4}-\frac{1}{2}b,\frac{3}{8}-\frac{1}{4}b,\frac{1}{8}+\frac{1}{4}b \\ \frac{9}{8}-\frac{3}{4}b,\frac{3}{8}+\frac{3}{4}b\end{matrix}\right] \\ &=\frac{1}{2\sqrt{2}\pi}\Gamma\left[\begin{matrix}\frac{3}{4}+\frac{1}{2}b,\frac{5}{4}-\frac{1}{2}b,\frac{3}{8}-\frac{1}{4}b,\frac{1}{8}+\frac{1}{4}b \\ \frac{9}{8}-\frac{3}{4}b,\frac{3}{8}+\frac{3}{4}b\end{matrix}\right].\end{aligned}$$

Now use $b=a+\frac{1}{2}$ to complete the proof. □

Theorem 3.5.

$$ {}_3F_2\left[\begin{matrix} 3a, a+\frac{4}{3}, -a-\frac{1}{3} \\ a+\frac{1}{3}, 2a+\frac{7}{6} \end{matrix} \middle| -\frac{1}{8}\right] = \left(\frac{2}{3}\right)^{3a} \Gamma\left[\begin{matrix} 2a+\frac{7}{6}, \frac{1}{2} \\ a+\frac{7}{6}, a+\frac{1}{2} \end{matrix}\right]. $$

$$ {}_3F_2\left[\begin{matrix} 3a, a+2, -a-1 \\ a, 2a+\frac{3}{2} \end{matrix} \middle| -\frac{1}{8}\right] = \left(\frac{2}{3}\right)^{3a} \Gamma\left[\begin{matrix} 2a+\frac{3}{2}, \frac{1}{2} \\ a+\frac{5}{6}, a+\frac{7}{6} \end{matrix}\right]. $$

Proof. Put $b = a+\frac{4}{3}$ and $b = a+2$ in (15). □

Lemma 3.6 (Nørlund). *Assume that all parameters have a real part* >0. *Put* $\sigma \equiv \alpha_0 + \sum_{k=1}^{n}(\alpha_k - \rho_k)$, $\mathrm{Re}\,\sigma > 0$. *Then*

$$ \lim_{\xi\to 1^-} (1-\xi)^{\sigma}\ {}_{n+1}F_n\left[\begin{matrix} \alpha_0, \ldots, \alpha_n \\ \rho_1, \ldots, \rho_n \end{matrix} \middle| \xi\right] = \Gamma\left[\begin{matrix} \rho_1, \ldots, \rho_n, \sigma \\ \alpha_0, \ldots, \alpha_n \end{matrix}\right]. \tag{18} $$

Proof. We note that the hypergeometric series is divergent. Furthermore this series has complex analytic continuation for $|\xi| > 1$. This is the reason why we take $\lim_{\xi\to 1^-}$. For $n=1$, this follows from the analytic continuation formula for ${}_2F_1$.

$$ {}_2F_1\left[\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} \middle| z\right] = \Gamma\left[\begin{matrix} \gamma, \gamma-\beta-\alpha \\ \gamma-\beta, \gamma-\alpha \end{matrix}\right] {}_2F_1\left[\begin{matrix} \alpha, \beta \\ \alpha+\beta+1-\gamma \end{matrix} \middle| 1-z\right] $$
$$ + \Gamma\left[\begin{matrix} \gamma, \alpha+\beta-\gamma \\ \alpha, \beta \end{matrix}\right] (1-z)^{\gamma-\alpha-\beta}\ {}_2F_1\left[\begin{matrix} \gamma-\alpha, \gamma-\beta \\ 1+\gamma-\alpha-\beta \end{matrix} \middle| 1-z\right]. \quad □ $$

The following formula was conjectured by Gosper.

Theorem 3.7.

$$ {}_4F_3\left[\begin{matrix} 3a, 1+a, b, 1-b \\ a, \frac{3a+b+1}{2}, \frac{3a-b+2}{2} \end{matrix} \middle| -\frac{1}{8}\right] = \Gamma\left[\begin{matrix} \frac{3a+b+1}{2}, \frac{3a-b+2}{2} \\ \frac{3a+1}{2}, \frac{3a+2}{2} \end{matrix}\right]. $$

Proof. For $x \to -\frac{1}{8}$, the RHS of (10) becomes $0\times\infty$, because the series is divergent. Therefore we shall use the limiting formula (18). Put

$$ \xi \equiv \frac{-27x}{(1-4x)^3} \ \Rightarrow\ 1-\xi = \frac{(1+8x)^2(1-x)}{(1-4x)^3}. $$

The limit $x \to -\frac{1}{8}$ for the RHS of (10) becomes

$$ \lim_{x\to-\frac{1}{8}} (1-4x)^{\frac{1}{2}-3a}(1-x)^{-\frac{1}{2}}(1-\xi)^{\frac{1}{2}}\ {}_3F_2\left[\begin{matrix} a+\frac{1}{3}, a+\frac{2}{3}, a+1 \\ \frac{1}{2}(3a+b+1), 1+\frac{1}{2}(3a-b) \end{matrix} \middle| \xi\right] $$

$$ \overset{\text{by}(18)}{=} \left(\frac{3}{2}\right)^{\frac{1}{2}-3a} \left(\frac{9}{8}\right)^{-\frac{1}{2}} \Gamma\left[\begin{matrix} \frac{3a+b+1}{2}, \frac{3a-b+2}{2} \\ a+\frac{1}{3}, a+\frac{2}{3}, a+1 \end{matrix}\right] $$

$$ \overset{\text{by}(52)}{=} \frac{3^{-\frac{1}{2}-3a}\sqrt{\pi}}{2^{-1-3a}3^{\frac{1}{2}-(3a+1)}2\pi} \Gamma\left[\begin{matrix} \frac{3a+b+1}{2}, \frac{3a-b+2}{2} \\ 3a+1 \end{matrix}\right] $$

$$ = 2^{3a}\Gamma\left[\begin{matrix} \frac{3a+b+1}{2}, \frac{3a-b+2}{2} \\ 3a+1, \frac{1}{2} \end{matrix}\right] \overset{\text{by}(50)}{=} \Gamma\left[\begin{matrix} \frac{3a+b+1}{2}, \frac{3a-b+2}{2} \\ \frac{3a+1}{2}, 1+\frac{3a}{2} \end{matrix}\right]. $$

This proves the theorem. □

Corollary 3.8.

$$-2\ {}_3F_2\left[\begin{matrix}3a,b,1-b\\ \frac{3a+b+1}{2},\frac{3a-b+2}{2}\end{matrix}\,\middle|\,-\frac{1}{8}\right]+3\ {}_3F_2\left[\begin{matrix}3a+1,b,1-b\\ \frac{3a+b+1}{2},\frac{3a-b+2}{2}\end{matrix}\,\middle|\,-\frac{1}{8}\right]=\Gamma\left[\begin{matrix}\frac{3a+b+1}{2},\frac{3a-b+2}{2}\\ \frac{3a+1}{2},\frac{3a+2}{2}\end{matrix}\right]. \tag{19}$$

$${}_3F_2\left[\begin{matrix}3a,b,1-b\\ \frac{3a+b+1}{2},\frac{3a-b+2}{2}\end{matrix}\,\middle|\,-\frac{1}{8}\right]-\frac{3}{2}\frac{b(1-b)}{(3a+b+1)(3a-b+2)}\ {}_3F_2\left[\begin{matrix}3a+1,b,1-b\\ \frac{3a+b+1}{2},\frac{3a-b+2}{2}\end{matrix}\,\middle|\,-\frac{1}{8}\right]=\Gamma\left[\begin{matrix}\frac{3a+b+1}{2},\frac{3a-b+2}{2}\\ \frac{3a+1}{2},\frac{3a+2}{2}\end{matrix}\right]. \tag{20}$$

Proof. Use formulas (55) and (56). □

Corollary 3.9.

$${}_3F_2\left[\begin{matrix}3b,b,1-b\\ 2b,b+\frac{1}{2}\end{matrix}\,\middle|\,-\frac{1}{8}\right]=\frac{1}{3}\Gamma\left[\begin{matrix}2b,b+\frac{1}{2}\\ \frac{3}{2}b,\frac{3}{2}b+\frac{1}{2}\end{matrix}\right]+\frac{2}{3}2^{3b-3}\left(\Gamma\left[\begin{matrix}\frac{1}{2}b,b+\frac{1}{2}\\ \frac{1}{2},\frac{3}{2}b\end{matrix}\right]\right)^2.$$

$${}_3F_2\left[\begin{matrix}3b,b+1,2-b\\ 2b+1,b+\frac{3}{2}\end{matrix}\,\middle|\,-\frac{1}{8}\right]=\frac{8(2b+1)}{3(b-1)}\left(\Gamma\left[\begin{matrix}2b,b+\frac{1}{2}\\ \frac{3}{2}b,\frac{3}{2}b+\frac{1}{2}\end{matrix}\right]-2^{3b-3}\left(\Gamma\left[\begin{matrix}\frac{1}{2}b,b+\frac{1}{2}\\ \frac{1}{2},\frac{3}{2}b\end{matrix}\right]\right)^2\right). \tag{21}$$

Proof. Put $a=b-\frac{1}{3}$ in (19) and (20) and use Champion's formula [3]. □

3.3. The case $x=\frac{1}{4}$.

Theorem 3.10.

$${}_4F_3\left[\begin{matrix}3a,1+a,b,1-b\\ a,\frac{3a+b+1}{2},\frac{3a-b+2}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right]=4^{a+\frac{1}{3}}\Gamma\left[\begin{matrix}\frac{3a+b+1}{2},\frac{3a-b+2}{2},a+\frac{1}{3}\\ 3a+1,\frac{a+b}{2}+\frac{1}{6},\frac{a-b}{2}+\frac{2}{3}\end{matrix}\right]. \tag{22}$$

$${}_3F_2\left[\begin{matrix}3a,b,1-b\\ \frac{3a+b+1}{2},\frac{3a-b+2}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right]=4^{a}\Gamma\left[\begin{matrix}\frac{3a+b+1}{2},\frac{3a-b+2}{2},a+1\\ 3a+1,\frac{a+b}{2}+\frac{1}{2},\frac{a-b}{2}+1\end{matrix}\right]. \tag{23}$$

$${}_3F_2\left[\begin{matrix}3a+1,b,1-b\\ \frac{3a+b+1}{2},\frac{3a-b+2}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right]=\frac{4^a}{3}\Gamma\left[\begin{matrix}\frac{3a+b+1}{2},\frac{3a-b+2}{2}\\ 3a+1\end{matrix}\right]\times\left[2^{\frac{2}{3}}\Gamma\left[\begin{matrix}a+\frac{1}{3}\\ \frac{a+b}{2}+\frac{1}{6},\frac{a-b}{2}+\frac{2}{3}\end{matrix}\right]+2\Gamma\left[\begin{matrix}a+1\\ \frac{a+b}{2}+\frac{1}{2},\frac{a-b}{2}+1\end{matrix}\right]\right]. \tag{24}$$

$$(25)\quad {}_3F_2\left[\begin{matrix} 3a+1,1+b,2-b \\ \frac{3a+b+3}{2},2+\frac{3a-b}{2} \end{matrix}\middle|\frac{1}{4}\right]=\frac{4^{a+1}}{3b(1-b)}\Gamma\left[\begin{matrix} \frac{3a+b+3}{2},2+\frac{3a-b}{2} \\ 3a+1 \end{matrix}\right] \times\left[2^{\frac{2}{3}}\Gamma\left[\begin{matrix} a+\frac{1}{3} \\ \frac{a+b}{2}+\frac{1}{6},\frac{a-b}{2}+\frac{2}{3} \end{matrix}\right]-\Gamma\left[\begin{matrix} a+1 \\ \frac{a+b}{2}+\frac{1}{2},\frac{a-b}{2}+1 \end{matrix}\right]\right].$$

Proof. Let $x\to\frac{1}{4}^-$ in Bailey's formula (9). Then the RHS equals

$$\lim_{x\to\frac{1}{4}^-}(1-4x)^{-3a}\,{}_3F_2\left[\begin{matrix} a,a+\frac{1}{3},a+\frac{2}{3} \\ \frac{1}{2}(3a+b+1),1+\frac{1}{2}(3a-b) \end{matrix}\middle|\frac{-27x}{(1-4x)^3}\right]$$
$$=\left(\frac{27}{4}\right)^{-a}\Gamma\left[\begin{matrix} \frac{1}{3},\frac{2}{3},\frac{3a+b+1}{2},\frac{3a-b+2}{2} \\ a+\frac{1}{3},a+\frac{2}{3},\frac{a+b+1}{2},\frac{a-b+2}{2} \end{matrix}\right]$$
$$\overset{\text{by}(52),(51)}{=}4^a\Gamma\left[\begin{matrix} \frac{3a+b+1}{2},\frac{3a-b+2}{2},a+1 \\ 3a+1,\frac{a+b+1}{2},\frac{a-b+2}{2} \end{matrix}\right].$$

Similarly, let $x\to\frac{1}{4}^-$ in RHS of (10):

$$\lim_{x\to\frac{1}{4}^-}(1+8x)(1-4x)^{-3a-1}\,{}_3F_2\left[\begin{matrix} a+1,a+\frac{1}{3},a+\frac{2}{3} \\ \frac{3a+b+1}{2},1+\frac{3a-b}{2} \end{matrix}\middle|\frac{-27x}{(1-4x)^3}\right]$$
$$=3\left(\frac{27}{4}\right)^{-a-\frac{1}{3}}\Gamma\left[\begin{matrix} \frac{1}{3},\frac{2}{3},\frac{3a+b+1}{2},\frac{3a-b+2}{2} \\ a+\frac{2}{3},a+1,\frac{a+b}{2}+\frac{1}{6},\frac{a-b}{2}+\frac{2}{3} \end{matrix}\right]$$
$$\overset{\text{by}(52),(51)}{=}4^{a+\frac{1}{3}}\Gamma\left[\begin{matrix} \frac{3a+b+1}{2},\frac{3a-b+2}{2},a+\frac{1}{3} \\ 3a+1,\frac{a+b}{2}+\frac{1}{6},\frac{a-b}{2}+\frac{2}{3} \end{matrix}\right].$$

Put this into formulas (9), (10), (13) and (14) to conclude the proof. □

The factor $2^{\frac{2}{3}}$ in formulas (24) and (25) can be removed:

Corollary 3.11.

$${}_3F_2\left[\begin{matrix} 3a+1,b,1-b \\ \frac{3a+b+1}{2},\frac{3a-b+2}{2} \end{matrix}\middle|\frac{1}{4}\right]=\frac{2^{3a}}{3}\Gamma\left[\begin{matrix} \frac{3a+b+1}{2},\frac{3a-b+2}{2} \\ 3a+1,\frac{1}{2} \end{matrix}\right] \times\left[\Gamma\left[\begin{matrix} \frac{a}{2}+\frac{1}{6},\frac{a}{2}+\frac{2}{3} \\ \frac{a+b}{2}+\frac{1}{6},\frac{a-b}{2}+\frac{2}{3} \end{matrix}\right]+2\Gamma\left[\begin{matrix} \frac{a}{2}+\frac{1}{2},\frac{a}{2}+1 \\ \frac{a+b}{2}+\frac{1}{2},\frac{a-b}{2}+1 \end{matrix}\right]\right].$$

$${}_3F_2\left[\begin{matrix} 3a+1,1+b,2-b \\ \frac{3a+b+3}{2},2+\frac{3a-b}{2} \end{matrix}\middle|\frac{1}{4}\right]=\frac{2^{3a+2}}{3b(1-b)}\Gamma\left[\begin{matrix} \frac{3a+b+3}{2},2+\frac{3a-b}{2} \\ 3a+1,\frac{1}{2} \end{matrix}\right] \times\left[\Gamma\left[\begin{matrix} \frac{a}{2}+\frac{1}{6},\frac{a}{2}+\frac{2}{3} \\ \frac{a+b}{2}+\frac{1}{6},\frac{a-b}{2}+\frac{2}{3} \end{matrix}\right]-\Gamma\left[\begin{matrix} \frac{a}{2}+\frac{1}{2},\frac{a}{2}+1 \\ \frac{a+b}{2}+\frac{1}{2},\frac{a-b}{2}+1 \end{matrix}\right]\right].$$

Proof. Use the Legendre duplication (50). □

The previous formulas for $x=\frac{1}{4}$ assume a special form in the terminating case.

Theorem 3.12.

$$
{}_4F_3\left[\begin{array}{c}-n,-\frac{n}{3}+1,n+2\beta-1,2-n-2\beta\\ -\frac{n}{3},\beta,\frac{3}{2}-n-\beta\end{array}\middle|\frac{1}{4}\right]
=\begin{cases}\dfrac{3\ (3N)!\ (-1)^N}{2^{2N}\ N!(\beta,\frac{1}{2}+\beta+2N)_N}, & n=3N+1,\\ 0,\ \text{where } n=3N+2, & \text{where } N\in\mathbb{N}_0.\end{cases}
$$

Proof. The LHS of formula (22) is a polynomial if $3a$, and not a, is a negative integer. Put $3a=-n\Rightarrow n=3N+1\vee n=3N+2$, $N\in\mathbb{N}_0$. When $3a=-(3N+2)$, we have $3a+1=-3N-1$, $a+\frac{1}{3}=-N-\frac{1}{3}$. The RHS is then zero. In order to deal with the case $3a=-(3N+1)$, the Gauss triplication formula may be used on the LHS. In the following, we put

$$b=2\beta+n-1=2\beta+3N,\ n=3N+1.$$

$$
\begin{aligned}
\text{LHS}&\overset{\text{by}(52)}{=}\frac{4^{-N}2\pi}{3^{-3N-\frac{1}{2}}}\Gamma\left[\begin{array}{c}\frac{1}{2}(b-3N),\frac{1}{2}(1-b-3N)\\ \frac{1}{2}(b-N),\frac{1}{2}(1-b-N),-N+\frac{1}{3},-N+\frac{2}{3}\end{array}\right]\\
&\overset{\text{by}(51)}{=}\frac{3^{3N}\sqrt{3}\ 2\pi}{2^{2N}}\Gamma\left[\begin{array}{c}\beta,\frac{1}{2}-\beta-3N,\frac{1}{3},\frac{2}{3}\\ \beta+N,\frac{1}{2}-\beta-2N,-N+\frac{1}{3},-N+\frac{2}{3}\end{array}\right]\frac{\sqrt{3}}{2\pi}\\
&=\frac{3^{3N+1}(\frac{1}{2}-\beta)_{-3N}}{2^{2N}(\beta)_N(\frac{1}{2}-\beta)_{-2N}(\frac{1}{3},\frac{2}{3})_{-N}}=\frac{3^{3N+1}(1,\frac{1}{3},\frac{2}{3})_N(\frac{1}{2}+\beta)_{2N}(-1)^N}{2^{2N}N!(\frac{1}{2}+\beta)_{3N}}.
\end{aligned}
$$

Finally, to conclude the proof [11, p. 22] can be used. □

Theorem 3.13.

$$
{}_3F_2\left[\begin{array}{c}-n,n+2\beta,1-n-2\beta\\ \frac{1}{2}+\beta,1-n-\beta\end{array}\middle|\frac{1}{4}\right]=\begin{cases}\dfrac{(3N)!\ (-1)^N}{2^{2N}\ N!(\beta+2N,\frac{1}{2}+\beta)_N}, & n=3N,\\ 0,\ \text{where } n\in\{1,2,4,5,\ldots\}.\end{cases}
$$

Proof. The LHS of formula (23) is a polynomial if $3a$ is a negative integer. Similarly, the only nonzero case is $3a=-3N$, for which the Gauss triplication formula may be used on the LHS. In the following, we put

$$b=2\beta+n=2\beta+3N,\ n=3N.$$

$$
\begin{aligned}
\text{LHS}&\overset{\text{by}(52),(51)}{=}\frac{4^{-N}}{3^{-3N}}\Gamma\left[\begin{array}{c}\frac{1}{2}(1+b-3N),\frac{1}{2}(2-b-3N),\frac{1}{3},\frac{2}{3}\\ \frac{1}{2}(1+b-N),\frac{1}{2}(2-b-N),-N+\frac{1}{3},-N+\frac{2}{3}\end{array}\right]\\
&=\frac{3^{3N}}{2^{2N}}\Gamma\left[\begin{array}{c}\frac{1}{2}+\beta,1-\beta-3N,\frac{1}{3},\frac{2}{3}\\ \frac{1}{2}+\beta+N,1-\beta-2N,-N+\frac{1}{3},-N+\frac{2}{3}\end{array}\right]\\
&=\frac{3^{3N}(1-\beta)_{-3N}}{(\frac{1}{2}+\beta)_N(1-\beta)_{-2N}(\frac{1}{3},\frac{2}{3})_{-N}2^{2N}}=\frac{3^{3N}(-1)^n(\beta)_{2N}(\frac{1}{3},\frac{2}{3},1)_N}{(1,\frac{1}{2}+\beta)_N(\beta)_{3N}2^{2N}}=\text{RHS}.
\end{aligned}
$$

Finally use [11, p. 22] to conclude the proof. □

Theorem 3.14.

$$
{}_3F_2\left[\begin{matrix}-n, n+2\beta, 1-n-2\beta\\ \beta, \frac{1}{2}-n-\beta\end{matrix}\,\middle|\,\frac{1}{4}\right]
= \begin{cases}\dfrac{(-1)^N\ (3N)!}{2^{2N}\ N!(\beta,\frac{1}{2}+\beta+2N)_N}, & n=3N,\\ 0, \text{ where } n=3N+1,\ N\in\mathbb{N}_0, \\ \dfrac{(-1)^{N+1}\ (3N+2)!}{2^{2N+1}\ N!(\beta,\frac{3}{2}+\beta+2N)_{N+1}}, & n=3N+2.\end{cases}
$$

Proof. The LHS of formula (24) is a polynomial if $3a+1$ is a negative integer. So, we let $3a+1=-n \Rightarrow a=-\frac{n+1}{3}$.

(1) When $3a=-(3N+1)$, we have $n=3N$. Then the first term survives, whereas the second is zero. Now using the Gauss triplication formula on the LHS with $b=2\beta+n=2\beta+3N$ we get:

$$
\text{LHS} \overset{\text{by}(52),(51)}{=}
$$

$$
\frac{4^{-N-\frac{1}{3}}2\pi\ 2^{1+\frac{2}{3}}}{3^{-3N+\frac{1}{2}}}\Gamma\left[\begin{matrix}\frac{1}{2}(b-3N), \frac{1}{2}(1-b-3N), \frac{1}{3}, \frac{2}{3}\\ \frac{1}{2}(b-N), \frac{1}{2}(1-b-N), -N+\frac{1}{3}, -N+\frac{2}{3}\end{matrix}\right]\frac{\sqrt{3}}{2\pi}
$$

$$
=\frac{3^{3N}}{2^{2N}}\Gamma\left[\begin{matrix}\beta, \frac{1}{2}-\beta-3N, \frac{1}{3}, \frac{2}{3}\\ \beta+N, \frac{1}{2}-\beta-2N, -N+\frac{1}{3}, -N+\frac{2}{3}\end{matrix}\right]
$$

$$
=\frac{3^{3N}(\frac{1}{2}-\beta)_{-3N}}{2^{2N}(\beta)_N(\frac{1}{2}-\beta)_{-2N}(\frac{1}{3},\frac{2}{3})_{-N}}=\frac{3^{3N}(1,\frac{1}{3},\frac{2}{3})_N(\frac{1}{2}+\beta)_{2N}(-1)^N}{2^{2N}N!(\frac{1}{2}+\beta)_{3N}(\beta)_N}=\text{RHS}.
$$

Finally, [11, p. 22] can be used to conclude the proof.

(2) When $3a=-(3N+2)$, we have $n=3N+1$, and both terms are zero.

(3) When $a=-N-1$, we have $n=3N+2$. In this case the first term is zero, but the second survives. Using $b=2\beta+n=2\beta+2+3N$, the Gauss triplication formula gives for the LHS:

$$
\text{LHS} \overset{\text{by}(52),(51)}{=} \frac{4^{-N}\ \pi}{3^{-3N-2+\frac{1}{2}}}\Gamma\left[\begin{matrix}\frac{1}{2}(b-3N-2), \frac{1}{2}(-1-b-3N), -\frac{1}{3}, -\frac{2}{3}\\ \frac{1}{2}(b-N), \frac{1}{2}(1-b-N), -N+\frac{1}{3}, -N+\frac{2}{3}\end{matrix}\right]\frac{1}{3\sqrt{3}\pi}
$$

$$
=\frac{3^{3N}}{2^{2N}}\Gamma\left[\begin{matrix}\beta, -\frac{3}{2}-\beta-3N, -\frac{1}{3}, -\frac{2}{3}\\ \beta+N+1, -\frac{1}{2}-\beta-2N, -N-\frac{1}{3}, -N-\frac{2}{3}\end{matrix}\right]
$$

$$
=\frac{3^{3N}(-\frac{1}{2}-\beta)_{-3N-1}}{2^{2N}(\beta)_{1+N}(-\frac{1}{2}-\beta)_{-2N}(-\frac{1}{3},-\frac{2}{3})_{-N}}=\frac{3^{3N}(1,\frac{4}{3},\frac{5}{3})_N(\frac{3}{2}+\beta)_{2N}(-1)^{N+1}}{2^{2N}N!(\frac{3}{2}+\beta)_{3N+1}(\beta)_{1+N}}
$$

$$
=\frac{(3)_{3N}(-1)^{N+1}}{2^{2N}N!(\frac{3}{2}+\beta+2N)_{N+1}(\beta)_{1+N}}=\text{RHS}.
$$

Finally, we may use [11, p. 22] to conclude the proof. □

Theorem 3.15.

$$
{}_3\mathrm{F}_2\left[\begin{matrix} -n, 2-\beta-n, 1+n+\beta \\ 1+\frac{1}{2}\beta, \frac{1}{2}(3-\beta)-n \end{matrix}\middle|\frac{1}{4}\right]
= \begin{cases} \dfrac{(-1)^N\ (3N)!\ \beta(1-\beta)}{2^{2N}\ N!(\beta+3N)(1-\beta-3N)} \dfrac{(\frac{1}{2}+\frac{1}{2}\beta)_{2N}}{(\frac{1}{2}\beta)_N(-\frac{1}{2}+\frac{1}{2}\beta)_{3N}}, & n=3N, \\ 0, \text{ where } n=3N+1,\ N\in\mathbb{N}_0, & \\ \dfrac{2^{1-2N}(-1)^N\ (4)_{3N}}{(\beta+2+3N)(1+\beta+3N)(2)_N} \dfrac{(\frac{3}{2}+\frac{1}{2}\beta)_{2N}}{(1+\frac{1}{2}\beta)_N(\frac{3}{2}+\frac{1}{2}\beta)_{3N}}, & n=3N+2. \end{cases}
$$

Proof. The LHS of formula (25) is a polynomial if $3a+1$ is a negative integer. Put $3a+1=-n \Rightarrow a=-\frac{n+1}{3}$.

When $3a=-(3N+1)$, we have $n=3N$. In this case the first term survives, whereas the second is zero. If we use $b=\beta+n=\beta+3N$, the Gauss triplication formula applied on the LHS gives:

$$
\begin{aligned}
\text{LHS} &\overset{\text{by(52),(51)}}{=} \frac{4^{-N+\frac{2}{3}}2^{1+\frac{2}{3}}\ \pi}{3^{-3N+\frac{1}{2}}b(1-b)}\Gamma\left[\begin{matrix} \frac{1}{2}(b-3N+2), \frac{1}{2}(3-b-3N), \frac{1}{3}, \frac{2}{3} \\ \frac{1}{2}(b-N), \frac{1}{2}(1-b-N), -N+\frac{1}{3}, -N+\frac{2}{3} \end{matrix}\right]\frac{\sqrt{3}}{2\pi} \\
&= \frac{3^{3N}4^{-N+1}}{(1-\beta-3N)(\beta+3N)}\Gamma\left[\begin{matrix} \frac{1}{2}\beta+1, -3N-\frac{1}{2}\beta+\frac{3}{2} \\ \frac{1}{2}\beta+N, \frac{1}{2}(1-\beta)-2N \end{matrix}\right]\left(\frac{1}{3}, \frac{2}{3}\right)_N \\
&= \frac{3^{3N}\left(\frac{1}{3}, \frac{2}{3}\right)_N\ 4^{-N-1}\left(\frac{1}{2}(3-\beta)\right)_{-3N}}{(1-\beta-3N)(\beta+3N)(\frac{1}{2}\beta)_N(\frac{1}{2}(1-\beta))_{-2N}}\frac{1}{2}\beta\frac{1}{2}(1-\beta) \\
&= \frac{3^{3N}(1, \frac{1}{3}, \frac{2}{3})_N(\frac{1}{2}(1+\beta))_{2N}\beta(1-\beta)(-1)^N}{4^N N!(\frac{1}{2}\beta)_N(-\frac{1}{2}+\frac{1}{2}\beta)_{3N}(1-\beta-3N)(\beta+3N)} = \text{RHS}.
\end{aligned}
$$

Finally use [11, p. 22] to conclude.

When $3a=-(3N+2)$, we have $n=3N+1$, and both terms are zero.

When $a=-N-1$, we have $n=3N+2$. Then the first term is zero and the second survives. Again, we apply the Gauss triplication formula on the LHS. Using $b=\beta+n=\beta+2+3N$ gives:

$$
\begin{aligned}
\text{LHS} &\overset{\text{by(52)}}{=} -\frac{2\pi\ 4^{-N}}{3^{-3N-\frac{3}{2}}b(1-b)}\Gamma\left[\begin{matrix} \frac{1}{2}(b-3N), \frac{1}{2}(1-b-3N) \\ \frac{1}{2}(b-N), \frac{1}{2}(1-b-N), -N-\frac{1}{3}, -N-\frac{2}{3} \end{matrix}\right] \\
&= -\frac{2\pi 3^{3N+\frac{3}{2}}4^{-N}}{(\beta+2+3N)(-1-\beta-3N)} \\
&\times\Gamma\left[\begin{matrix} 1+\frac{1}{2}\beta, -\frac{1}{2}-\frac{1}{2}\beta-3N, -\frac{2}{3}, -\frac{1}{3} \\ \frac{1}{2}\beta+N+1, -\frac{1}{2}-\frac{1}{2}\beta-2N, -N-\frac{1}{3}, -N-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \end{matrix}\right] \\
&\overset{\text{by(51)}}{=} -\frac{2\pi\ 3^{3N+\frac{3}{2}}4^{-N}(-\frac{1}{2}-\frac{1}{2}\beta)_{-3N}}{(\beta+2+3N)(-1-\beta-3N)(1+\frac{1}{2}\beta)_N(-\frac{1}{2}-\frac{1}{2}\beta)_{-2N}(-\frac{1}{3}, -\frac{2}{3})_{-N}}\frac{1}{3\sqrt{3}\pi} \\
&= \frac{2^{1-2N}3^{3N}(2, \frac{4}{3}, \frac{5}{3})_N(\frac{3}{2}+\frac{1}{2}\beta)_{2N}(-1)^N}{(2)_N(\beta+2+3N)(1+\beta+3N)(\frac{3}{2}+\frac{1}{2}\beta)_{3N}(1+\frac{1}{2}\beta)_N} = \text{RHS}.
\end{aligned}
$$

Finally, to conclude the proof we use [11, p. 22]. □

Theorem 3.16.

$$ {}_4F_3\left[\begin{matrix} -3n-1, -n+\frac{2}{3}, 2\beta+n, 1-2\beta-n \\ -n-\frac{1}{3}, \beta-n, \frac{1}{2}-\beta-2n \end{matrix}\middle| \frac{1}{4}\right] = \frac{3^{3n+1}\left(\frac{1}{3},\frac{2}{3}\right)_n}{2^{2n}\left(1-\beta,\frac{1}{2}+\beta+n\right)_n}, \; n\in\mathbb{N}_0, $$

$$ {}_4F_3\left[\begin{matrix} -3n-2, -n+\frac{1}{3}, 2\beta+n, 1-2\beta-n \\ -n-\frac{2}{3}, \beta-n-\frac{1}{2}, -\beta-2n \end{matrix}\middle| \frac{1}{4}\right] = 0, \; n\in\mathbb{N}_0. $$

Proof. Consider the terminating case for the companion formula (10). The degree of the RHS is called n and we take $\lim_{x\to\frac{1}{4}}$. The product of $(1-4x)^{-1-3a}$ and the highest order term involves the factor $(1-4x)^{-1-3a-3n}$. Then there are three possibilities:

(1) $a+\frac{1}{3}=-n$: Term no. n has limit $\neq 0$, the other terms go to zero.

(2) $a+\frac{2}{3}=-n$: All terms go to zero.

(3) $a+1=-n$: All terms go to zero, but the left hand side has an undefined limit.

For $a=-n-\frac{1}{3}$ the RHS equals

$$ 3\frac{(-n,-n+\frac{1}{3},-n+\frac{2}{3})_n(-27)^n}{(\frac{b-3n}{2},(\frac{1-b-3n}{2})_n\; n!4^n} = \frac{3(-1)^n\; n!(-1)^n 3^{3n}}{\left(\frac{1}{3},\frac{2}{3}\right)_{-n}(\frac{b-3n}{2},\frac{1-b-3n}{2})_n\; n!2^{2n}}. $$

Now put $b=2\beta+n$ to conclude the proof. □

Theorem 3.17.

$$ {}_3F_2\left[\begin{matrix} 3a+1,1,2 \\ \frac{3a+3}{2},\frac{3a+4}{2} \end{matrix}\middle| \frac{1}{4}\right] = \frac{(3a+1)(3a+2)}{6}\Psi\left[\begin{matrix} \frac{a}{2}+\frac{2}{3},\frac{a}{2}+\frac{1}{2} \\ \frac{a}{2}+\frac{1}{6},\frac{a}{2}+1 \end{matrix}\right]. $$

Proof. We compute the limit $b\to 0$ in (25). The Legendre duplication on the LHS gives

$$ {}_3F_2\left[\begin{matrix} 3a+1,1+b,2-b \\ \frac{3a+b+3}{2},2+\frac{3a-b}{2} \end{matrix}\middle| \frac{1}{4}\right] = \frac{2^{3a+2}}{3b(1-b)}\Gamma\left[\begin{matrix} \frac{3a+b+3}{2},2+\frac{3a-b}{2} \\ 3a+1,\frac{1}{2} \end{matrix}\right] $$
$$ \times\left(\Gamma\left[\begin{matrix} \frac{a}{2}+\frac{1}{6},\frac{a}{2}+\frac{2}{3} \\ \frac{a}{2}+\frac{1}{6}+\frac{b}{2},\frac{a}{2}+\frac{2}{3}-\frac{b}{2} \end{matrix}\right] - \Gamma\left[\begin{matrix} \frac{a}{2}+\frac{1}{2},\frac{a}{2}+1 \\ \frac{a}{2}+\frac{1}{2}+\frac{b}{2},\frac{a}{2}+1-\frac{b}{2} \end{matrix}\right]\right). $$

To compute the limit $b\to 0$, use $\Gamma(x+h)\cong\Gamma(x)(1+k\Psi(x-1))$. The LHS equals

$$\begin{aligned}&\frac{2^{3a+2}}{3b(1-b)}\Gamma\left[\begin{matrix}\frac{3a+3}{2},2+\frac{3a}{2}\\ 3a+1,\frac{1}{2}\end{matrix}\right]\\ &\quad\times\left(\frac{1}{(1+\frac{b}{2}\Psi(\frac{a}{2}-\frac{5}{6}))(1-\frac{b}{2}\Psi(\frac{a}{2}-\frac{1}{3}))}-\frac{1}{(1+\frac{b}{2}\Psi(\frac{a}{2}-\frac{1}{2}))(1-\frac{b}{2}\Psi(\frac{a}{2}))}\right)\\ &\cong\frac{2^{3a+1}}{3\cdot 2^{3a}}\Gamma\left[\begin{matrix}\frac{3a+3}{2},2+\frac{3a}{2}\\ \frac{1}{2}3a+1,\frac{3a}{2}+1\end{matrix}\right]\\ &\quad\times\left(-\Psi\left(\frac{a}{2}+\frac{1}{6}\right)+\Psi\left(\frac{a}{2}+\frac{2}{3}\right)+\Psi\left(\frac{a}{2}+\frac{1}{2}\right)-\Psi\left(\frac{a}{2}+1\right)\right)=\text{RHS}.\end{aligned}$$

□

4. Bailey's second cubic transformation

4.1. Statements of the transformations. Karlsson [7] has previously studied cubic hypergeometric transformations of the kind: $G(x)=P(x)H(z)$, where G, H are Clausen hypergeometric functions, and come to the conclusion that the only transformations of interest are those for which

$$z=\frac{27x^2}{(4-x)^3}.$$

We therefore repeat Bailey's second cubic transformation [2, (4.06)], which reads:

$$(26)\quad {}_3F_2\left[\begin{matrix}3a,b,c\\ 2b,2c\end{matrix}\,\middle|\,x\right]\cong\left(1-\frac{1}{4}x\right)^{-3a}{}_3F_2\left[\begin{matrix}a,a+\frac{1}{3},a+\frac{2}{3}\\ b+\frac{1}{2},c+\frac{1}{2}\end{matrix}\,\middle|\,\frac{27x^2}{(4-x)^3}\right].$$

Proof. This can be carried out by equating the coefficients for powers of x and applying the Saalschütz summation formula. □

Remark 2. This formula is given by Bailey in the quoted paper. It is of equal interest as Bailey's first cubic transformation. If $x=1$, then the cubic function $z=1$. The function argument on the RHS lies between 0 and 1 for $-1<x<1$, so convergence is assured.

For this formula to be true, we need to restrict the parameters by putting $3a=b+c-\frac{1}{2}$ as follows.

Theorem 4.1 (Gessel & Stanton [6, (5.6)]).

$$(27)\quad\begin{aligned}&{}_3F_2\left[\begin{matrix}b+c-\frac{1}{2},b,c\\ 2b,2c\end{matrix}\,\middle|\,x\right]\\ &=\left(1-\frac{1}{4}x\right)^{-b-c+\frac{1}{2}}{}_3F_2\left[\begin{matrix}\frac{1}{3}(b+c)-\frac{1}{6},\frac{1}{3}(b+c)+\frac{1}{6},\frac{1}{3}(b+c)+\frac{1}{2}\\ b+\frac{1}{2},c+\frac{1}{2}\end{matrix}\,\middle|\,\frac{27x^2}{(4-x)^3}\right].\end{aligned}$$

Remark 3. This formula is given with a typo in [1, p. 185].

Gessel & Stanton [6, (5.7)] stated the following companion.

$$
\begin{aligned}
&{}_4\mathrm{F}_3\left[\begin{matrix} b+c-\frac{1}{2}, \frac{2}{3}(b+c+1), b, c \\ \frac{2}{3}(b+c)-\frac{1}{3}, 2b, 2c \end{matrix}\,\middle|\, x\right] \\
&= \left(1+\frac{1}{8}x\right)\left(1-\frac{1}{4}x\right)^{-b-c-\frac{1}{2}} \\
&\times {}_3\mathrm{F}_2\left[\begin{matrix} \frac{1}{3}(b+c)+\frac{1}{6}, \frac{1}{3}(b+c)+\frac{5}{6}, \frac{1}{3}(b+c)+\frac{1}{2} \\ b+\frac{1}{2}, c+\frac{1}{2} \end{matrix}\,\middle|\, \frac{27x}{(4-x)^3}\right].
\end{aligned}
\tag{28}
$$

Put $3a=b+c-\frac{1}{2}$ in (26) to get $s=-\frac{1}{2}$. In contrast, $s=\frac{1}{2}$ in formula (28). We look for the inverse of these two transformations.

Theorem 4.2 (A new transformation).

$$
\begin{aligned}
&{}_3\mathrm{F}_2\left[\begin{matrix} b+c-\frac{1}{2}, b, c \\ 2b, 2c \end{matrix}\,\middle|\, x\right] \\
&= \left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2b-2c} {}_3\mathrm{F}_2\left[\begin{matrix} b+c-\frac{1}{2}, b, c \\ 2b, 2c \end{matrix}\,\middle|\, \frac{-8x}{\left(1+\sqrt{1-x}\right)^3}\right].
\end{aligned}
\tag{29}
$$

The following proof is primary, and the second proof of (47) is alternate.

Proof. For the inverse transformation, put

$$z=\frac{27x^2}{(4-x)^3} \Rightarrow z\,(4-x)^3-27x^2=0.$$

x is one of the two roots of this equation, which are zero for $z=0$, according to Goursat. From

$$\frac{27x_1^2}{(4-x_1)^3}=\frac{27x_2^2}{(4-x_2)^3}=z,$$

we have

$$
\begin{aligned}
&(64-48x_2+12x_2^2-x_2^3)x_1^2=(64-48x_1+12x_1^2-x_1^3)x_2^2 \Rightarrow \\
&x_1^2x_2^2(x_1-x_2)-48x_1x_2(x_1-x_2)+64(x_1^2-x_2^2)=0 \Rightarrow \\
&x_1^2x_2^2-48x_1x_2+64(x_1+x_2)=0 \Rightarrow \\
&x_1^2x_2^2+(64-48x_1)x_2+64x_1=0 \Rightarrow \\
&x_2=\frac{1}{2x_1^2}\left(48x_1-64+16(4-x_1)\sqrt{1-x_1}\right) \\
&\quad=\frac{8}{x_1^2}\left(3x_1-4+16(4-x_1)\sqrt{1-x_1}\right) \\
&\quad=\frac{8}{x_1^2}\left(-1+\sqrt{1-x_1}\right)^3=\frac{-8x_1}{(1+\sqrt{1-x_1})^3}.
\end{aligned}
$$

This means that we can replace x by $\frac{-8x_1}{1+\sqrt{1-x_1}^3}$ in formulas (26) and (28). □

For brevity, put

$$\xi_1 \equiv \sqrt{1-x_1},\ -\frac{\pi}{2} \le \operatorname{Arg}\xi_1 \le \frac{\pi}{2},\ x_2 = \frac{-8(1-\xi_1^2)}{(1-\xi_1)^3} = \frac{-8(1-\xi_1)}{(1-\xi_1)^2}.$$

Region of validity: clearly, x must not be on the cut.

Theorem 4.3. *Consider then the new transformations*

$$\begin{aligned} x_2 - 1 &= \frac{-8+8\xi_1-1-2\xi_1-\xi_1^2}{(1+\xi_1)^2} = -\frac{9-6\xi_1+\xi_1^2}{(1+\xi_1)^2} \\ &= -\left(\frac{3-\xi_1}{1+\xi_1}\right)^2 \equiv u^2 \in [0,\infty). \end{aligned}$$

The boundary is given by

$$\operatorname{Re} x = \frac{-8(1-3u^2)}{(1+u^2)^2},\ \operatorname{Im} x = \frac{8(3u-u^3)}{(1+u^2)^2},\ -\sqrt{3} \le u \le \sqrt{3},$$

and, additionally, the line segment $[1,4]$ *on the real axis.*

Proof. Thus

$$\begin{aligned} &\frac{3-\xi_1}{1+\xi_1} = iu,\ u \in \mathbb{R}, \\ &\xi_1 = \frac{3-iu}{1+iu} = \frac{(3-iu)(1-iu)}{1+u^2} = \frac{3-4iu-u^2}{1+u^2}. \end{aligned}$$

Since $\operatorname{Re}\xi_1 \ge 0$, we can have only $u^2 \le 3$.

$$\begin{aligned} x_1 = (1-\xi_1)(1+\xi_1) &= \frac{-2+2iu}{1+iu}\frac{4}{1+iu} \\ &= \frac{-8(1-iu)^3}{(1+u^2)^2} = \frac{-8(1-3iu-3u^2+iu^3)}{(1+u^2)^2}. \end{aligned}$$

□

We next show that the same boundary applies to (26) and (28). For this we must solve the following equation:

$$\frac{27x^2}{(4-x)^3} = 1+t,\ t \in [0,\infty].$$

We first prove that the curve already found is correct. The line segment in parametric form is given by:

$$x = 4-\tau,\ \tau \in [0,3],\ \frac{27x^2}{(4-x)^3} = \frac{27}{\tau^3}(4-\tau)^2,$$

decreasing from $+\infty$ to 1. The curve is given by:

$$x = \frac{-8(1-iu)}{(1+iu)^2}, \; -\sqrt{3} \le u \le \sqrt{3},$$

$$4-x = \frac{1}{(1+iu)^2}(4+8iu-4u^2+8-8iu) = \frac{12-4u^2}{(1+iu)^2} = \frac{4(8-u^2)}{(1+iu)^2}.$$

Finally, z_2 is given by

$$\begin{aligned} z_2 &= \frac{27x^2}{(4-x)^3} = \frac{27\cdot 64(1-iu)^2(1+iu)^6}{(1+iu)^2 64(3-u^2)^3} \\ &= \frac{27(1+u^2)^2}{(3(1-\frac{1}{3}u^2))^3} = \frac{(1+u^2)^2}{(1-\frac{1}{3}u^2)^3} \in [0,\infty) \; \text{(twice)}. \end{aligned}$$

Theorem 4.4.

$$\begin{aligned} &{}_3\mathrm{F}_2\left[\begin{matrix} 3a+1, b, 3a-b+\frac{1}{2} \\ 2b, 6a-2b+1 \end{matrix} \middle| x\right] \\ &= \frac{1}{3}\left(1-\frac{1}{4}x\right)^{-3a} {}_3\mathrm{F}_2\left[\begin{matrix} a, a+\frac{1}{3}, a+\frac{2}{3} \\ b+\frac{1}{2}, 3a-b+1 \end{matrix} \middle| \frac{27x^2}{(4-x)^3}\right] \\ &+ \frac{2}{3}\left(1+\frac{1}{8}x\right)\left(1-\frac{1}{4}x\right)^{-1-3a} {}_3\mathrm{F}_2\left[\begin{matrix} a+\frac{1}{3}, a+\frac{2}{3}, a+1 \\ b+\frac{1}{2}, 3a-b+1 \end{matrix} \middle| \frac{27x^2}{(4-x)^3}\right]. \end{aligned} \tag{30}$$

Theorem 4.5.

$$\begin{aligned} &{}_3\mathrm{F}_2\left[\begin{matrix} 3a+1, b+1, 3a-b+\frac{3}{2} \\ 2b+1, 6a-2b+2 \end{matrix} \middle| x\right] \\ &= \frac{8}{3x}\left(1-\frac{1}{4}x\right)^{-3a}\left[\frac{1+\frac{1}{8}x}{1-\frac{1}{4}x} {}_3\mathrm{F}_2\left[\begin{matrix} a+\frac{1}{3}, a+\frac{2}{3}, a+1 \\ b+\frac{1}{2}, 3a-b+1 \end{matrix} \middle| \frac{27x^2}{(4-x)^3}\right]\right. \\ &\left. - {}_3\mathrm{F}_2\left[\begin{matrix} a, a+\frac{1}{3}, a+\frac{2}{3} \\ b+\frac{1}{2}, 3a-b+1 \end{matrix} \middle| \frac{27x^2}{(4-x)^3}\right]\right]. \end{aligned} \tag{31}$$

Proof. By the two contiguity relations (55) and (56), we obtain from (28):

$$\begin{aligned} &-\frac{1}{2}\,{}_3\mathrm{F}_2\left[\begin{matrix} 3a, b, 3a-b+\frac{1}{2} \\ 2b, 6a-2b+1 \end{matrix} \middle| x\right] + \frac{3}{2}\,{}_3\mathrm{F}_2\left[\begin{matrix} 3a+1, b, 3a-b+\frac{1}{2} \\ 2b, 6a-2b+1 \end{matrix} \middle| x\right] \\ &= \left(1-\frac{1}{4}x\right)^{-3a} {}_3\mathrm{F}_2\left[\begin{matrix} a, a+\frac{1}{3}, a+\frac{2}{3} \\ b+\frac{1}{2}, c+\frac{1}{2} \end{matrix} \middle| \frac{27x^2}{(4-x)^3}\right] \\ &= \left(1+\frac{1}{8}x\right)\left(1-\frac{1}{4}x\right)^{-1-3a} {}_3\mathrm{F}_2\left[\begin{matrix} a+\frac{1}{3}, a+\frac{2}{3}, a+1 \\ b+\frac{1}{2}, 3a-b+1 \end{matrix} \middle| \frac{27x}{(4-x)^3}\right] \\ &= {}_3\mathrm{F}_2\left[\begin{matrix} 3a, b, 3a-b+\frac{1}{2} \\ 2b, 6a-2b+1 \end{matrix} \middle| x\right] + \frac{3x}{8}\,{}_3\mathrm{F}_2\left[\begin{matrix} 3a+1, b, 3a-b+\frac{3}{2} \\ 2b+1, 6a-2b+2 \end{matrix} \middle| x\right]. \end{aligned}$$

Now multiply the first formula by $\frac{2}{3}$ and the second by $\frac{8}{3x}$ together with (26), to obtain formulas (30) and (31). $\square$

4.2. The case $x=-8$. Consider the case $c=3a-b+\frac{1}{2}$ in (26).

Theorem 4.6.

$$ {}_3F_2\left[\begin{matrix} 3a, a+\frac{1}{3}, 2a+\frac{1}{6} \\ 2a+\frac{2}{3}, 4a+\frac{1}{3}\end{matrix}\,\middle|\,-8\right] = 3^{-3a}\left(\Gamma\left[\begin{matrix} \frac{1}{2}, a+\frac{5}{6} \\ \frac{1}{2}(1+a), \frac{1}{2}a+\frac{5}{6}\end{matrix}\right]\right)^2. $$

Proof. Put $b=\frac{1}{3}+a$ and use Watson's summation formula to get

$$ \text{LHS} = 3^{-3a}\ {}_3F_2\left[\begin{matrix} a, a+\frac{1}{3}, a+\frac{2}{3} \\ a+\frac{5}{6}, 2a+\frac{2}{3}\end{matrix}\right] \overset{\text{by[5, p. 83 (3.55)]}}{=} \text{RHS}. $$

$\square$

Theorem 4.7.

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, b, 1-b \\ 2b, 2-2b\end{matrix}\,\middle|\,-8\right] = \frac{\pi}{\sqrt{3}}\Gamma\left[\begin{matrix} \frac{1}{2}+b, \frac{3}{2}-b \\ \frac{5}{6}-\frac{1}{2}b, \frac{1}{3}+\frac{1}{2}b, \frac{7}{6}-\frac{1}{2}b, \frac{2}{3}+\frac{1}{2}b\end{matrix}\right]. $$

Proof. Put $a=\frac{1}{6}$ and use Whipple's summation formula to get

$$ \text{LHS} = 3^{-3a}\ {}_3F_2\left[\begin{matrix} \frac{1}{6}, \frac{1}{2}, \frac{5}{6} \\ b+\frac{1}{2}, \frac{3}{2}-b\end{matrix}\right] = \text{RHS}. $$

$\square$

Theorem 4.8.

$$ {}_3F_2\left[\begin{matrix} 3a, a-\frac{1}{2}, 2a+1 \\ 2a-1, 4a+2\end{matrix}\,\middle|\,-8\right] = 3^{-3a}\Gamma\left[\begin{matrix} 2a+\frac{3}{2}, \frac{1}{2} \\ a+\frac{7}{6}, a+\frac{5}{6}\end{matrix}\right]. $$

$$ {}_3F_2\left[\begin{matrix} 3a, a-\frac{1}{6}, 2a+\frac{2}{3} \\ 2a-\frac{1}{3}, 4a+\frac{4}{3}\end{matrix}\,\middle|\,-8\right] = 3^{-3a}\Gamma\left[\begin{matrix} 2a+\frac{7}{6}, \frac{1}{2} \\ a+\frac{7}{6}, a+\frac{1}{2}\end{matrix}\right]. $$

$$ {}_2F_1\left[\begin{matrix} 3a, a+\frac{1}{6} \\ 4a+\frac{2}{3}\end{matrix}\,\middle|\,-8\right] = 3^{-3a}\Gamma\left[\begin{matrix} 2a+\frac{5}{6}, \frac{1}{2} \\ a+\frac{5}{6}, a+\frac{1}{2}\end{matrix}\right]. $$

Proof. Put

$$ b=a-\frac{1}{2},\ c=2a+1;\ b=a-\frac{1}{6},\ c=2a+\frac{2}{3};\ b=a+\frac{1}{6},\ c=2a+\frac{1}{3} $$

in (26) and use the Gauss summation formula. $\square$

Theorem 4.9.

$$ (32)\qquad {}_4F_3\left[\begin{matrix} b, c, b+c-\frac{1}{2}, \frac{2}{3}(b+c+1) \\ 2b, 2c, \frac{2}{3}(b+c-\frac{1}{2})\end{matrix}\,\middle|\,-8\right] = \frac{1}{2}\Gamma\left[\begin{matrix} b+\frac{1}{2}, c+\frac{1}{2} \\ \frac{1}{2}, b+c+\frac{1}{2}\end{matrix}\right]. $$

Proof. For $x\to -8$, the RHS of (28) becomes $0\times\infty$, because the series is divergent. Therefore we shall use the limit formula (18). Put

$$ \xi \equiv \frac{27x^2}{(4-x)^3} \Rightarrow\ 1-\xi = \frac{(1+\frac{1}{8}x)^2(1-x)}{\left(1-\frac{1}{4}x\right)^3}. $$

The limit $x \to -8$ for the RHS of (28) becomes, with $\sigma = \frac{1}{2}$:

$$\lim_{x\to -8}\left(1-\frac{1}{4}x\right)^{\frac{1}{2}-3a}(1-x)^{-\frac{1}{2}}(1-\xi)^{\frac{1}{2}}\,{}_3\mathrm{F}_2\left[\begin{matrix} a+\frac{1}{3},a+\frac{2}{3},a+1 \\ b+\frac{1}{2},c+\frac{1}{2}\end{matrix}\Bigg|\xi\right]$$
$$\overset{\text{by}(18)}{=} 3^{\frac{1}{2}-3a}\,\frac{1}{3}\Gamma\left[\begin{matrix} b+\frac{1}{2},c+\frac{1}{2},\frac{1}{2} \\ a+\frac{1}{3},a+\frac{2}{3},a+1\end{matrix}\right]$$
$$\overset{\text{by}(52)}{=} \frac{3^{-\frac{1}{2}-3a}\sqrt{\pi}}{3^{\frac{1}{2}-(3a+1)}\cdot 2\pi}\Gamma\left[\begin{matrix} b+\frac{1}{2},c+\frac{1}{2} \\ 3a+1\end{matrix}\right] = \frac{1}{2\sqrt{\pi}}\Gamma\left[\begin{matrix} b+\frac{1}{2},c+\frac{1}{2} \\ b+c+\frac{1}{2}\end{matrix}\right].$$

This proves the theorem. □

Corollary 4.10. *Put*

$$A \equiv \Gamma\left[\begin{matrix} a+\frac{5}{6},2a+\frac{2}{3} \\ \frac{1}{2},3a+1\end{matrix}\right],\ B \equiv 3^{-3a}\left(\Gamma\left[\begin{matrix} a+\frac{5}{6},\frac{1}{2} \\ \frac{1}{2}(a+1),\frac{1}{2}a+\frac{5}{6}\end{matrix}\right]\right)^2.$$

Then we have

$${}_3\mathrm{F}_2\left[\begin{matrix} 3a+1,a+\frac{1}{3},2a+\frac{1}{6} \\ 4a+\frac{1}{3},2a+\frac{2}{3}\end{matrix}\Bigg|-8\right] = \frac{1}{3}(A+B).$$
$${}_3\mathrm{F}_2\left[\begin{matrix} 3a+1,a+\frac{4}{3},2a+\frac{7}{6} \\ 4a+\frac{4}{3},2a+\frac{5}{3}\end{matrix}\Bigg|-8\right] = \frac{1}{3}\left(-\frac{1}{2}A+B\right).$$

Proof. We expand formula (32) to get

$$-\frac{1}{2}\,{}_3\mathrm{F}_2\left[\begin{matrix} b,c,b+c-\frac{1}{2} \\ 2b,2c\end{matrix}\Bigg|-8\right]+\frac{3}{2}\,{}_3\mathrm{F}_2\left[\begin{matrix} b,c,b+c+\frac{1}{2} \\ 2b,2c\end{matrix}\Bigg|-8\right]$$
$$=\frac{1}{2}\Gamma\left[\begin{matrix} b+\frac{1}{2},c+\frac{1}{2} \\ \frac{1}{2},b+c+\frac{1}{2}\end{matrix}\right].$$
$${}_3\mathrm{F}_2\left[\begin{matrix} b,c,b+c-\frac{1}{2} \\ 2b,2c\end{matrix}\Bigg|-8\right]-3\,{}_3\mathrm{F}_2\left[\begin{matrix} b+1,c+1b+c+\frac{1}{2} \\ 2b+1,2c+1\end{matrix}\Bigg|-8\right]$$
$$=\frac{1}{2}\Gamma\left[\begin{matrix} b+\frac{1}{2},c+\frac{1}{2} \\ \frac{1}{2},b+c+\frac{1}{2}\end{matrix}\right].$$

The first ${}_3\mathrm{F}_2$ is evaluated by putting $b=a+\frac{1}{3},\ c=2a+\frac{1}{6}$. Thus by Clausen's formula

$$-3^{-3a}\left(\Gamma\left[\begin{matrix} a+\frac{5}{6},\frac{1}{2} \\ \frac{1}{2}(a+1),\frac{1}{2}a+\frac{5}{6}\end{matrix}\right]\right)^2+3{}_3\mathrm{F}_2\left[\begin{matrix} 3a+1,a+\frac{1}{3},2a+\frac{1}{6} \\ 4a+\frac{1}{3},2a+\frac{2}{3}\end{matrix}\Bigg|-8\right]$$
$$=\Gamma\left[\begin{matrix} a+\frac{5}{6},2a+\frac{2}{3} \\ \frac{1}{2},3a+1\end{matrix}\right],$$
$$3^{-3a}\left(\Gamma\left[\begin{matrix} a+\frac{5}{6},\frac{1}{2} \\ \frac{1}{2}(a+1),\frac{1}{2}a+\frac{5}{6}\end{matrix}\right]\right)^2-3{}_3\mathrm{F}_2\left[\begin{matrix} 3a+1,a+\frac{4}{3},2a+\frac{7}{6} \\ 4a+\frac{4}{3},2a+\frac{5}{3}\end{matrix}\Bigg|-8\right]$$
$$=\Gamma\left[\begin{matrix} a+\frac{5}{6},2a+\frac{2}{3} \\ \frac{1}{2},3a+1\end{matrix}\right],$$

which completes the proof. □

Corollary 4.11.

$$ {}_3F_2\left[\begin{matrix} a+\frac{1}{6}, 3a, 2a+1 \\ 4a+\frac{2}{3}, 2a \end{matrix} \middle| -8\right] = \frac{1}{2}\Gamma\left[\begin{matrix} a+\frac{2}{3}, 2a+\frac{5}{6} \\ \frac{1}{2}, 3a+1 \end{matrix}\right]. \tag{33} $$

$$ {}_3F_2\left[\begin{matrix} 1, b, \frac{1}{2}-b \\ 2b, 1-2b \end{matrix} \middle| -8\right] = \frac{1}{3}\left(1+\Gamma\left[\begin{matrix} b+\frac{1}{2}, 1-b \\ \frac{1}{2}, 1 \end{matrix}\right]\right). \tag{34} $$

$$ {}_3F_2\left[\begin{matrix} b, b-\frac{1}{2}, \frac{2}{3}(b+1) \\ 2b, \frac{2}{3}b-\frac{1}{3} \end{matrix} \middle| -8\right] = 0. \tag{35} $$

Proof. To prove (33), put $b=a+\frac{1}{6}$, $c=2a+\frac{1}{3}$ in (32). To prove (34), put $a=0$. To prove (35), put $c=0$ in (32). □

4.3. The case $x=1$. Consider the case $c=3a-b+\frac{1}{2}$, $x=1$ in (26). That is

$$ {}_3F_2\left[\begin{matrix} 3a, b, 3a-b+\frac{1}{2} \\ 2b, 6a-2b+1 \end{matrix}\right] = \left(\frac{3}{4}\right)^{-3a} {}_3F_2\left[\begin{matrix} a, a+\frac{1}{3}, a+\frac{2}{3} \\ b+\frac{1}{2}, 3a-b+1 \end{matrix}\right]. \tag{36} $$

Corollary 4.12.

$$ {}_3F_2\left[\begin{matrix} 3a, b, 3a-b+\frac{1}{2} \\ 2b, 6a-2b+1 \end{matrix} \middle| -8\right] = 2^{-6a}\, {}_3F_2\left[\begin{matrix} 3a, b, 3a-b+\frac{1}{2} \\ 2b, 6a-2b+1 \end{matrix}\right]. \tag{37} $$

Proof. Put $x=-8$ into (26) and combine with (36). □

There are no reduction formulas for (37). Now apply Dixon's, Watson's and Whipple's summation formulas to (37). The results are then always $a=\frac{1}{6}$, $b=\frac{1}{2}$. This results in

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -8\right] = \frac{1}{2}\Gamma\left[\begin{matrix} \frac{5}{4}, \frac{1}{4} \\ \frac{3}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4} \end{matrix}\right]. $$

4.4. The limit $x \to 4$ for a polynomial of degree n. The previous formulas for $x \to \frac{1}{4}$ assume a special form in the terminating case.

Theorem 4.13.

$$ {}_3F_2\left[\begin{matrix} -n, \frac{1}{2}-b-n, b \\ 2b, 1-2b-2n \end{matrix} \middle| 4\right] = \begin{cases} 0, & n \not\equiv 0 \bmod 3 \\ \dfrac{(\frac{1}{3}, \frac{2}{3}, \frac{b}{2}, \frac{b+1}{2})_N}{(\frac{1}{2}+b, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3})_N}, & n=3N. \end{cases} $$

Proof. The highest power on the RHS of formula (26) is $\left(\frac{1}{4-x}\right)^{3N}$; it is exactly $\left(1-\frac{1}{4}x\right)^{3a}$, i.e. $a=-N$.

The polynomials with $a+\frac{1}{3}=-N$ have highest power $(1-x)^{3a+1}$. The polynomials with $a+\frac{2}{3}=-N$ have highest power $(1-x)^{3a+2}$. Both these cases have limit $\lim_{x\to 1}=0$.

For $a=-N$ we get

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}-3N,\frac12-b-3N,b\\2b,1-2b-6N\end{matrix}\,\middle|\,4\right]=\left(\frac{27}{4}\right)^N\frac{(-N,\ -N+\frac13,\ -N+\frac23)_N}{(\frac12+b,\ 1,\ 1-b-3N)_N}\\
&=\left(\frac{27}{4}\right)^N\frac{(-1)^N(1-b)_{-3N}}{(\frac13,\frac23)_{-N}(b+\frac12)_N(1-b)_{-2N}}=\left(\frac{27}{4}\right)^N\frac{(\frac13,\frac23)_N(b)_{2N}}{(b+\frac12)_N(b)_{3N}}\\
&=\frac{(\frac13,\ \frac23,\ \frac b2,\ \frac{b+1}{2})_N}{(\frac12+b,\ \frac b3,\ \frac{b+1}{3},\ \frac{b+2}{3})_N}.
\end{aligned}\tag{38}
$$

□

Corollary 4.14.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}-n,1-2b-n,n+2b\\b+\frac12,1-b-n\end{matrix}\,\middle|\,\frac14\right]\\
&=\begin{cases}0,\ n\not\equiv 0 \bmod 3\\ (-\frac14)^n\dfrac{(2b,\ 1-2b-2n)_n}{(b,\frac12-b-n)_n}\dfrac{(\frac13,\ \frac23,\ \frac b2,\ \frac{b+1}{2})_N}{(\frac12+b,\ \frac b3,\ \frac{b+1}{3},\ \frac{b+2}{3})_N},\ n=3N.\end{cases}
\end{aligned}
$$

Proof. Reverse the series (38). □

Theorem 4.15.

$$
{}_4F_3\left[\begin{matrix}-3N-1,-2N+\frac13,b,-3N-\frac12-b\\-2N-\frac23,2b,-6n-1-2b\end{matrix}\,\middle|\,4\right]=\frac{3^{3N+1}(\frac13,\frac23)_N(1+b)_{2N}}{2^{2N+1}(\frac12+b)_N(1+b)_{3N}}.\tag{39}
$$

Proof. The companion (28) can be used for $-3a-1=3N$. Then the RHS has the limit

$$
\frac32\left(\frac{27}{4}\right)^N\frac{(-N,-N+\frac13,-N+\frac23)_N}{(1,b+\frac12,-3N-b)_N}=\frac{3^{3N+1}(-1)^N(-b)_{-3N}}{2^{2N+1}(\frac12+b)_N(\frac23,\frac13)_{-N}(-b)_{-2N}}.
$$

□

Formula (39) can also be reversed.

5. A third cubic transformation

5.1. Statements of the transformations.

Theorem 5.1.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}a,b,a+b-\frac12\\2a,2b\end{matrix}\,\middle|\,x\right]=\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac12-a-b}\\
&\times{}_3F_2\left[\begin{matrix}a+b-\frac12,a-b+\frac12,b-a+\frac12\\a+\frac12,b+\frac12\end{matrix}\,\middle|\,\frac{-x^2}{8\left(1+\sqrt{1-x}\right)^3}\right].
\end{aligned}\tag{40}
$$

Proof. By formula (27),

$$
\begin{aligned}
&{}_3\mathrm{F}_2\left[\begin{matrix} a^\star, b, a^\star+b-\frac{1}{2} \\ 2b, 2a^\star \end{matrix}\,\middle|\, x\right] \\
&= \left(1-\frac{1}{4}x\right)^{\frac{1}{2}-a^\star-b} {}_3\mathrm{F}_2\left[\begin{matrix} \frac{1}{3}(a^\star+b)-\frac{1}{6}, \frac{1}{3}(a^\star+b)+\frac{1}{6}, \frac{1}{3}(a^\star+b)+\frac{1}{2} \\ b+\frac{1}{2}, a^\star+\frac{1}{2} \end{matrix}\,\middle|\, \frac{27x^2}{(4-x)^3}\right].
\end{aligned}
$$

In Bailey's first cubic transformation (9), put

$$x \to z,\ b \to b^\star - a^\star + \frac{1}{2},\ a \to \frac{1}{3}(a^\star+b^\star)-\frac{1}{6}$$

to obtain

$$
\begin{aligned}
&{}_3\mathrm{F}_2\left[\begin{matrix} \frac{1}{3}(a^\star+b^\star)-\frac{1}{6}, \frac{1}{3}(a^\star+b^\star)+\frac{1}{6}, \frac{1}{3}(a^\star+b^\star)+\frac{1}{2} \\ b^\star+\frac{1}{2}, a^\star+\frac{1}{2} \end{matrix}\,\middle|\, \frac{-27z}{(1-4z)^3}\right] \\
&= (1-4z)^{a^\star+b^\star-\frac{1}{2}}\ {}_3\mathrm{F}_2\left[\begin{matrix} a^\star+b^\star-\frac{1}{2}, a^\star-b^\star+\frac{1}{2}, b^\star-a^\star+\frac{1}{2} \\ a^\star+\frac{1}{2}, b^\star+\frac{1}{2}, \end{matrix}\,\middle|\, z\right].
\end{aligned}
$$

Now drop the $\star$, put

$$\frac{x^2}{(4-x)^3} = \frac{-z}{(1-4z)^3} \tag{41}$$

in the above formulas and solve for ${}_3\mathrm{F}_2\left[\begin{matrix} a, b, a+b-\frac{1}{2} \\ 2b, 2a \end{matrix}\,\middle|\, x\right]$. This gives

$$
\begin{aligned}
&{}_3\mathrm{F}_2\left[\begin{matrix} a, b, a+b-\frac{1}{2} \\ 2a, 2b \end{matrix}\,\middle|\, x\right] \\
&= \left(1-\frac{1}{4}x\right)^{\frac{1}{2}-a-b}(1-4z)^{a+b-\frac{1}{2}}\ {}_3\mathrm{F}_2\left[\begin{matrix} a+b-\frac{1}{2}, a-b+\frac{1}{2}, b-a+\frac{1}{2} \\ a+\frac{1}{2}, b+\frac{1}{2} \end{matrix}\,\middle|\, z\right].
\end{aligned} \tag{42}
$$

Formula (41) implies that $x=0$ must imply $z=0$. Next multiply to obtain

$$
\begin{aligned}
& x^2(1-12z+48z^2-64z^3) = -z(64-48x+12x^2-x^3) \\
& \Longrightarrow 0 = z(x^3+48x-64)+x^2(64z^3-48z^2-1) \\
& \qquad = x^2(xz-1)+48xz(1-xz)+64z(x^2z^2-1) \\
& \qquad = (xz-1)(x^2-48xz+64z(xz+1)).
\end{aligned}
$$

Since $xz=0$ is impossible, we have to solve the quadratic equation (43) for z in terms of x, and are bound to choose the positive root.

$$64xz^2-(48x-64)z+x^2 = 0. \tag{43}$$

The solution of (43) is

$$
\begin{aligned}
z &= \frac{1}{2\cdot 64x}\left(48x-64\pm 16(4-x)\sqrt{1-x}\right)\\
&= \frac{1}{8x}\left(3x-4\,\overset{+}{(-)}\,(4-x)\sqrt{1-x}\right) \qquad (44)\\
&= -\frac{1}{8x}\left(1-\sqrt{1-x}\right)^3.
\end{aligned}
$$

Or

$$
z=\frac{-x^2}{8\left(1+\sqrt{1-x}\right)^3},\ \operatorname{Arg}\sqrt{1-x}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right).
$$

Finally,

$$
1-4z=1-\frac{1}{2x}(3x-4+(4-x)\sqrt{1-x})=\frac{4-x}{2x}\left(1-\sqrt{1-x}\right)
$$

and so

$$
\frac{1-\frac{1}{4}x}{1-4z}=\frac{1}{4}\frac{4-x}{1-4z}=\frac{1}{4}\frac{2x}{1-\sqrt{1-x}}=\frac{1+\sqrt{1-x}}{2}.
$$

Put this into (42) to finish the proof. □

Theorem 5.2 (A companion to (40)).

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,b,a+b-\frac{1}{2},\frac{2}{3}(a+b+1)\\ 2a,2b,\frac{2}{3}(a+b-\frac{1}{2})\end{matrix}\,\middle|\, x\right]\\
&= \left(\frac{1+\sqrt{1-x}}{2}\right)^{-\frac{1}{2}-a-b}\frac{x+\frac{1}{8}x^2}{4x-4+(4-x)\sqrt{1-x}} \qquad (45)\\
&{}_4F_3\left[\begin{matrix} a+b-\frac{1}{2},a-b+\frac{1}{2},b-a+\frac{1}{2},\frac{1}{3}(a+b)+\frac{5}{6}\\ a+\frac{1}{2},b+\frac{1}{2},\frac{1}{3}(a+b)-\frac{1}{6}\end{matrix}\,\middle|\,\frac{-x^2}{8\left(1+\sqrt{1-x}\right)^3}\right].
\end{aligned}
$$

Proof. In Bailey's second companion (28), put

$$
c\to 3a-b+\frac{1}{2}\to a^\star,\ 3a\to a^\star+b-\frac{1}{2}.
$$

In Bailey's first companion (10), put

$$
x\to z,\ b\to b^\star-a^\star+\frac{1}{2},\ a\to\frac{1}{3}(a^\star+b^\star)-\frac{1}{6}.
$$

Now drop the $\star$, use (41) and continue as in the previous proof. □

We have $s=-\frac{1}{2}$ on both sides of the two theorems. Hence, both transformations are valid for the principal branch of the logarithm.

5.2. The case $x = 1$, $z = -\frac{1}{8}$. Consider the case $a+b=1$. By Whipple's theorem we obtain

$$
{}_3F_2\left[\begin{matrix} a, 1-a, \frac{1}{2} \\ 2a, 2(1-a) \end{matrix}\right] = \Gamma\left[\begin{matrix} a, a+\frac{1}{2}, 1-a, \frac{3}{2}-a \\ \frac{3}{2}a, \frac{1}{2}(a+1), 1-\frac{1}{2}a, \frac{3}{2}(1-a) \end{matrix}\right]
= \frac{2^{-1}}{\pi}\Gamma\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}(1-a), \frac{1}{2}+a, \frac{3}{2}-a \\ \frac{3}{2}a, \frac{3}{2}(1-a) \end{matrix}\right],
$$

which implies

$$
{}_3F_2\left[\begin{matrix} \frac{1}{2}, 2a-\frac{1}{2}, \frac{3}{2}-2a \\ a+\frac{1}{2}, \frac{3}{2}-a \end{matrix} \middle| -\frac{1}{8}\right] = \frac{1}{2\sqrt{2}\pi}\Gamma\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}(1-a), \frac{1}{2}+a, \frac{3}{2}-a \\ \frac{3}{2}a, \frac{3}{2}(1-a) \end{matrix}\right].
$$

5.3. The companion in the case $x = 1$. This is a limit again.

Theorem 5.3.

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a+b-\frac{1}{2}, a-b+\frac{1}{2}, b-a+\frac{1}{2}, \frac{1}{3}(a+b)+\frac{5}{6} \\ a+\frac{1}{2}, b+\frac{1}{2}, \frac{1}{3}(a+b)-\frac{1}{6} \end{matrix} \middle| -\frac{1}{8}\right] \\
&= 2^{a+b-\frac{1}{2}}\Gamma\left[\begin{matrix} a+\frac{1}{2}, b+\frac{1}{2} \\ \frac{1}{2}, a+b+\frac{1}{2} \end{matrix}\right].
\end{aligned} \tag{46}
$$

Proof. Multiply (45) by $\sqrt{1-x}$. For $x \to 1$, the LHS becomes $0 \times \infty$, because the series is divergent. By (18), the LHS becomes

$$
\begin{aligned}
&\lim_{x\to 1}\left(\sqrt{1-x}\,{}_4F_3\left[\begin{matrix} a, b, a+b-\frac{1}{2}, \frac{2}{3}(a+b+1) \\ 2a, 2b, \frac{2}{3}(a+b-\frac{1}{2}) \end{matrix} \middle| x\right]\right) \\
&= \Gamma\left[\begin{matrix} 2a, 2b, \frac{2}{3}(a+b-\frac{1}{2}), \frac{1}{2} \\ a, b, a+b-\frac{1}{2}, \frac{2}{3}(a+b+1) \end{matrix}\right] \\
&= \frac{3}{2}\Gamma\left[\begin{matrix} 2a, 2b, \frac{1}{2} \\ a, b, a+b+\frac{1}{2} \end{matrix}\right] \overset{\text{by}(50)}{=} 3\cdot 2^{2a+2b-3}\Gamma\left[\begin{matrix} a+\frac{1}{2}, b+\frac{1}{2} \\ \frac{1}{2}, a+b+\frac{1}{2} \end{matrix}\right].
\end{aligned}
$$

The RHS multiplied by $\sqrt{1-x}$ turns to

$$
\frac{3}{8}\left(\frac{1}{2}\right)^{-\frac{1}{2}-a-b}{}_4F_3\left[\begin{matrix} a+b-\frac{1}{2}, a-b+\frac{1}{2}, b-a+\frac{1}{2}, \frac{1}{3}(a+b)+\frac{5}{6} \\ a+\frac{1}{2}, b+\frac{1}{2}, \frac{1}{3}(a+b)-\frac{1}{6} \end{matrix} \middle| -\frac{1}{8}\right].
$$

Finally, solve for

$$
{}_4F_3\left[\begin{matrix} a+b-\frac{1}{2}, a-b+\frac{1}{2}, b-a+\frac{1}{2}, \frac{1}{3}(a+b)+\frac{5}{6} \\ a+\frac{1}{2}, b+\frac{1}{2}, \frac{1}{3}(a+b)-\frac{1}{6} \end{matrix} \middle| -\frac{1}{8}\right].
$$

□

5.4. The companion in the case $x = -8$.

Theorem 5.4.

$$
{}_4F_3\left[\begin{matrix} a, b, a+b-\frac{1}{2}, \frac{2}{3}(a+b+1) \\ 2a, 2b, \frac{2}{3}(a+b-\frac{1}{2}) \end{matrix} \middle| -8\right] = \frac{1}{2}\Gamma\left[\begin{matrix} a+\frac{1}{2}, b+\frac{1}{2} \\ \frac{1}{2}, a+b+\frac{1}{2} \end{matrix}\right].
$$

Proof. Consider the singular limit $x \to -8 (= \frac{0}{0})$ on the RHS. By L'Hôpital's rule we find that

$$\lim_{x\to -8} \frac{x+\frac{1}{8}x^2}{4x-4+(4-x)\sqrt{1-x}} = \lim_{x\to -8} \frac{1+\frac{1}{4}x}{4-\sqrt{1-x}-(4-x)\frac{1}{2\sqrt{1-x}}} = 1.$$

Finally use (46). □

6. Another proof of the second cubic transformation

We present here an alternative proof of the second cubic transformation (29).

6.1. Statement of the transformation.

Theorem 6.1.

$$(47)\quad \begin{aligned} &{}_3\mathrm{F}_2\left[\begin{matrix} a,b,a+b-\frac{1}{2} \\ 2a,2b \end{matrix}\,\middle|\, x\right] \\ &= \left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2a-2b} {}_3\mathrm{F}_2\left[\begin{matrix} a,b,a+b-\frac{1}{2} \\ 2a,2b \end{matrix}\,\middle|\, \frac{-8x}{\left(1+\sqrt{1-x}\right)^3}\right]. \end{aligned}$$

Proof. Consider the rational function $R(x)$ of degree 3 that appears in Bailey's first cubic transformation formulas, which expresses a doubly parametrized ${}_3\mathrm{F}_2(R(x))$ in terms of another ${}_3\mathrm{F}_2(x)$. By solving the algebraic equation $R_1(x_1) = R_2(x_2)$ (and ignoring the trivial solution $x_1 = x_2$) one finds an algebraic relationship between x_1, x_2; and if x_1, x_2 are so related, then Bailey's formula immediately yields a formula that relates ${}_3\mathrm{F}_2(x_1)$ and ${}_3\mathrm{F}_2(x_2)$.

We start with the third cubic transformation (40) and put

$$\xi_i \equiv \sqrt{1-x_i}, \text{ with } \operatorname{Arg}\xi_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

From (44) we obtain

$$-8z = \frac{x^2}{(1+\sqrt{1-x})^3} = \frac{(1-\xi^2)^2}{(1+\xi)^3} = \frac{(1-\xi)^2}{1+\xi},$$

where $\xi \equiv \xi_1 \vee \xi_2$.

ξ_i is one of the two roots, which must therefore satisfy

$$\frac{(1-\xi_1)^2}{1+\xi_1} = -8z = \frac{(1-\xi_2)^2}{1+\xi_2}.$$

That is

$$\begin{aligned} &(1+\xi_2)\left(1-2\xi_1+\xi_1^2\right) = (1+\xi_1)\left(1-2\xi_2+\xi_2^2\right) \Rightarrow \\ &1-2\xi_1+\xi_1^2+\xi_2-2\xi_1\xi_2+\xi_1^2\xi_2 = 1-2\xi_2+\xi_2^2+\xi_1-2\xi_1\xi_2+\xi_1\xi_2^2. \end{aligned}$$

Or

$$0 = 3\xi_1 - 3\xi_2 + \xi_2^2 - \xi_1^2 + \xi_1\xi_2(\xi_2-\xi_1) = (\xi_1-\xi_2)(3-(\xi_1+\xi_2)-\xi_1\xi_2).$$

Since $\xi_1 \neq \xi_2$, we have $\xi_1+\xi_2+\xi_1\xi_2=3$. Or

$$\xi_2 = \frac{3-\xi_1}{1+\xi_1},\ 1-\xi_2 = \frac{-2(1-\xi_1)}{1+\xi_1},$$
$$1+\xi_2 = \frac{4}{1+\xi_1},\ \frac{1+\xi_1}{1+\xi_2} = \left(\frac{1+\xi_1}{2}\right)^2.$$

We can now compute x_2 in terms of x_1:

$$x_2 = 1-\xi_2^2 = \frac{-8(1-\xi_1)}{(1+\xi_1)^2} = \frac{-8x_1}{\left(1+\sqrt{1-x_1}\right)^3}.$$

□

Theorem 6.2 (A companion to (47)).

$$\begin{aligned}&{}_4\mathrm{F}_3\left[\begin{matrix} a,b,a+b-\frac{1}{2},\frac{2}{3}(a+b+1) \\ 2a,2b,\frac{2}{3}(a+b-\frac{1}{2})\end{matrix}\,\middle|\, x\right]\\ &= \frac{3-\sqrt{1-x}}{2\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-2a-2b} \qquad (48)\\ &\times {}_3\mathrm{F}_2\left[\begin{matrix} a,b,a+b-\frac{1}{2},\frac{2}{3}(a+b+1) \\ 2a,2b,\frac{2}{3}(a+b-\frac{1}{2})\end{matrix}\,\middle|\, \frac{-8x}{\left(1+\sqrt{1-x}\right)^3}\right].\end{aligned}$$

6.2. The case $x=-1$. Then we have

$$x_2 = -\frac{8}{x^2}\left(1-\sqrt{1-x}\right)^3 = 8\left(\sqrt{2}-1\right)^3 = 8\left(5\sqrt{2}-7\right).$$

According to [10, §7.45], no formula with free parameters is possible. The list has a supplement (82a).

$$\begin{aligned}4\log\frac{1+\sqrt{2}}{2} &= {}_3\mathrm{F}_2\left[\begin{matrix} 1,1,\frac{3}{2} \\ 2,2\end{matrix}\,\middle|\, -1\right]\\ &= \left(\frac{1+\sqrt{2}}{2}\right)^{-3} {}_3\mathrm{F}_2\left[\begin{matrix} 1,1,\frac{3}{2} \\ 2,2\end{matrix}\,\middle|\, 8(5\sqrt{2}-7)\right].\end{aligned}$$

That is

$${}_3\mathrm{F}_2\left[\begin{matrix} 1,1,\frac{3}{2} \\ 2,2\end{matrix}\,\middle|\, 8(5\sqrt{2}-7)\right] = \frac{5\sqrt{2}+7}{2}\log\frac{1+\sqrt{2}}{2}.$$

6.3. The case $x=1$.

(1) $a=b=\frac{1}{2}$

$${}_3\mathrm{F}_2\left[\begin{matrix} \frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 1,1\end{matrix}\,\middle|\, -8\right] = \frac{1}{8\pi^3}\left(\Gamma\left(\frac{1}{4}\right)\right)^4.$$

(2) $a=b=1$

$${}_3\mathrm{F}_2\left[\begin{matrix} 1,1,\frac{3}{2} \\ 2,2\end{matrix}\,\middle|\, -8\right] = \frac{1}{2}\log 2.$$

(3) $b = 2a - \frac{1}{2}$

$$ {}_3F_2\left[\begin{matrix} a, 2a-\frac{1}{2}, 3a-1 \\ 2a, 4a-1 \end{matrix}\,\middle|\, -8\right] = \frac{1}{4}\left(\Gamma\left[\begin{matrix} a+\frac{1}{2}, \frac{1}{2}a \\ \frac{1}{2}, \frac{3}{2}a \end{matrix}\right]\right)^2. $$

Proof. Use the Clausen, Gauss, Watson and twice Legendre duplication formulas.

$$ \text{LHS (47)} = \left({}_2F_1\left[\begin{matrix} \frac{1}{2}a, \frac{3}{2}a-\frac{1}{2} \\ 2a \end{matrix}\right]\right)^2 = \left(\Gamma\left[\begin{matrix} 2a, \frac{1}{2}, \\ \frac{3}{2}a, \frac{1}{2}(a+1) \end{matrix}\right]\right)^2. $$

$$ \text{LHS } = 2^{2-6a}\left(\Gamma\left[\begin{matrix} 2a, \frac{1}{2}, \\ \frac{3}{2}a, \frac{1}{2}(a+1) \end{matrix}\right]\right)^2 = 2^{-2a}\left(\Gamma\left[\begin{matrix} a, a+\frac{1}{2} \\ \frac{3}{2}a, \frac{1}{2}(a+1) \end{matrix}\right]\right)^2. $$

We use the Legendre duplication twice in the end. □

(4) $b = 1 - a$

$$ {}_3F_2\left[\begin{matrix} a, 1-a, \frac{1}{2} \\ 2a, 2-2a \end{matrix}\,\middle|\, -8\right] = \frac{1}{2}\Gamma\left[\begin{matrix} a, a+\frac{1}{2}, 1-a, \frac{3}{2}-a \\ \frac{3}{2}a, \frac{1}{2}a+\frac{1}{2}, -\frac{1}{2}a+1, \frac{3}{2}-\frac{3}{2}a \end{matrix}\right] $$

Proof. Use the Whipple formula. □

6.4. Companion: Reduction to lower order. The method is similar to cases discussed under the third transformation.

(1) $b = 2a$

$$ {}_3F_2\left[\begin{matrix} a, 3a-\frac{1}{2}, 2a+\frac{2}{3} \\ 4a, 2a-\frac{1}{3} \end{matrix}\,\middle|\, x\right] $$
$$ = \frac{3-\sqrt{1-x}}{2\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{1-6a} {}_3F_2\left[\begin{matrix} a, 3a-\frac{1}{2}, 2a+\frac{2}{3} \\ 4a, 2a-\frac{1}{3} \end{matrix}\,\middle|\, \frac{-8x}{\left(1+\sqrt{1-x}\right)^3}\right]. $$

(2) $b = a + \frac{1}{2}$ leads to the trivial formula

$$ {}_3F_2\left[\begin{matrix} a, a+\frac{1}{2}, \frac{4}{3}a+1 \\ 2a+1, \frac{4}{3}a \end{matrix}\,\middle|\, x\right] $$
$$ = \frac{3-\sqrt{1-x}}{2\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{-3a} {}_3F_2\left[\begin{matrix} a, a+\frac{1}{2}, \frac{4}{3}a+1 \\ 2a+1, \frac{4}{3}a \end{matrix}\,\middle|\, \frac{-8x}{\left(1+\sqrt{1-x}\right)^3}\right]. $$

(3) $b = 2a - 1$

$$ {}_2F_1\left[\begin{matrix} a, 3a-\frac{3}{2} \\ 4a-2 \end{matrix}\,\middle|\, x\right] $$
$$ = \frac{3-\sqrt{1-x}}{2\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{3-6a} {}_2F_1\left[\begin{matrix} a, 3a-\frac{3}{2} \\ 4a-2 \end{matrix}\,\middle|\, \frac{-8x}{\left(1+\sqrt{1-x}\right)^3}\right]. $$

The Euler transformation on RHS gives

$$
\begin{aligned}
&{}_2\mathrm{F}_1\left[\begin{matrix} a, 3a-\frac{3}{2} \\ 4a-2 \end{matrix}\,\middle|\, x\right] \\
&= \frac{1}{\sqrt{1-x}}\left(\frac{1+\sqrt{1-x}}{2}\right)^{4-6a} {}_2\mathrm{F}_1\left[\begin{matrix} a-\frac{1}{2}, 3a-2 \\ 4a-2 \end{matrix}\,\middle|\, \frac{-8x}{(1+\sqrt{1-x})^3}\right].
\end{aligned}
$$

Finally, the Euler transformation on RHS gives

$$
\begin{aligned}
&{}_2\mathrm{F}_1\left[\begin{matrix} a-\frac{1}{2}, 3a-2 \\ 4a-2 \end{matrix}\,\middle|\, x\right] \\
&= \left(\frac{1+\sqrt{1-x}}{2}\right)^{4-6a} {}_2\mathrm{F}_1\left[\begin{matrix} a-\frac{1}{2}, 3a-2 \\ 4a-2 \end{matrix}\,\middle|\, \frac{-8x}{(1+\sqrt{1-x})^3}\right].
\end{aligned}
$$

6.5. The case $x=1$ for companion identity.

Corollary 6.3.

$$
{}_4\mathrm{F}_3\left[\begin{matrix} a, b, a+b-\frac{1}{2}, \frac{2}{3}(a+b+1) \\ 2a, 2b, \frac{2}{3}(a+b-\frac{1}{2}) \end{matrix}\,\middle|\, -8\right] = \frac{1}{2}\Gamma\left[\begin{matrix} a+\frac{1}{2}, b+\frac{1}{2} \\ \frac{1}{2}, a+b+\frac{1}{2} \end{matrix}\right].
$$

Proof. Multiply (48) by $\sqrt{1-x}$. For $x\to 1$, the LHS becomes $0\times\infty$, because the series is divergent. Therefore, we shall use the limiting formula (18) with $\sigma=\frac{1}{2}$. The LHS of (48) becomes

$$
\Gamma\left[\begin{matrix} 2a, 2b, \frac{2}{3}(a+b-\frac{1}{2}), \frac{1}{2} \\ a, b, a+b-\frac{1}{2}, \frac{2}{3}(a-b-\frac{1}{2})+1 \end{matrix}\right] = \frac{1}{\frac{2}{3}(a+b-\frac{1}{2})} 2^{2a+2b-2}\Gamma\left[\begin{matrix} a+\frac{1}{2}, b+\frac{1}{2} \\ \frac{1}{2}, a+b-\frac{1}{2} \end{matrix}\right].
$$

While the RHS becomes, for $x=1$

$$
\frac{3}{2}2^{2a+2b-1}\, {}_4\mathrm{F}_3\left[\begin{matrix} a, b, a+b-\frac{1}{2}, \frac{2}{3}(a+b+1) \\ 2a, 2b, \frac{2}{3}(a+b-\frac{1}{2}) \end{matrix}\,\middle|\, -8\right].
$$

This proves the theorem. □

Corollary 6.4. *Two further* ${}_3\mathrm{F}_2(-8)$ *formulas:*

$$
\begin{aligned}
&{}_3\mathrm{F}_2\left[\begin{matrix} a, 2a-\frac{1}{2}, 3a \\ 2a, 4a-1 \end{matrix}\,\middle|\, -8\right] \\
&= \frac{1}{3}\Gamma\left[\begin{matrix} a+\frac{1}{2}, 2a \\ \frac{1}{2}, 3a \end{matrix}\right] + \frac{1}{12}\left(\Gamma\left[\begin{matrix} a+\frac{1}{2}, \frac{1}{2}a \\ \frac{1}{2}, \frac{3}{2}a \end{matrix}\right]\right)^2. \\
&{}_3\mathrm{F}_2\left[\begin{matrix} a+1, 2a+\frac{1}{2}, 3a \\ 2a+1, 4a \end{matrix}\,\middle|\, -8\right] \\
&= -\frac{1}{6}\Gamma\left[\begin{matrix} a+\frac{1}{2}, 2a \\ \frac{1}{2}, 3a \end{matrix}\right] + \frac{1}{12}\left(\Gamma\left[\begin{matrix} a+\frac{1}{2}, \frac{1}{2}a \\ \frac{1}{2}, \frac{3}{2}a \end{matrix}\right]\right)^2.
\end{aligned}
$$

Proof. Let $b=2a-\frac{1}{2}$, and write ${}_4\mathrm{F}_3(-8)$ in terms of ${}_3\mathrm{F}_2(-8)$. Then, using previous results, we obtain

$$\Gamma\left[\begin{array}{c} a+\frac{1}{2},2a \\ \frac{1}{2},3a \end{array}\right]=-\frac{1}{4}\left(\Gamma\left[\begin{array}{c} a+\frac{1}{2},\frac{1}{2}a \\ \frac{1}{2},\frac{3}{2}a \end{array}\right]\right)^2+3\ {}_3\mathrm{F}_2\left[\begin{array}{c} a,2a-\frac{1}{2},3a \\ 2a,4a-1 \end{array}\middle|-8\right];$$

$$\frac{1}{2}\Gamma\left[\begin{array}{c} a+\frac{1}{2},2a \\ \frac{1}{2},3a \end{array}\right]=\frac{1}{4}\left(\Gamma\left[\begin{array}{c} a+\frac{1}{2},\frac{1}{2}a \\ \frac{1}{2},\frac{3}{2}a \end{array}\right]\right)^2-3\ {}_3\mathrm{F}_2\left[\begin{array}{c} a+1,2a+\frac{1}{2},3a \\ 2a+1,4a \end{array}\middle|-8\right].$$

Finally, solve for the two hypergeometric functions. □

7. Conclusion

We have proved many new hypergeometric formulas, which can be used in the theory of multiple hypergeometric series. Many of the quoted formulas appear without proofs or in other form in the literature. Apparently, this theory is not fully developed, and many similar results will most certainly appear henceforth.

8. Discussion

One could look for a “fifth” cubic transformation, emanating from Bailey’s first cubic transformation. Perhaps by attempting to solve $R_1(x_1)=R_1(x_2)$, and obtain an additional algebraic map $x_1\mapsto x_2$ on which a transformation of a 2-parameter ${}_3\mathrm{F}_2$ could be based?

9. Appendix

A formula for the Pochhammer symbol [11, p. 22 (2)]:

$$(a)_{kn}=k^{nk}\prod_{m=0}^{k-1}\left(\frac{a+m}{k}\right)_n.$$

Definition 1. The generalized Γ function is defined as follows:

$$\Gamma\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_r \end{array}\right]\equiv\frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_r)}. \tag{49}$$

The formula (49) is called balanced, if

$$\sum_{k=1}^{p}a_k=\sum_{k=1}^{r}b_k,\quad p=r.$$

Definition 2. The Gauss Ψ function or the Digamma function is defined by

$$\Psi(z)\equiv\mathrm{D}\log(\Gamma(z+1)).$$

Definition 3. The generalized Ψ function is defined as follows:

$$\Psi\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_r \end{array}\right]\equiv\sum_{k=1}^{p}\Psi(a_k)-\sum_{k=1}^{r}\Psi(b_k).$$

Some examples of balanced Γ functions formulas are:

(1) The Legendre duplication formula

$$\Gamma(2x)=\frac{2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})}{\sqrt{\pi}}. \tag{50}$$

(2) The Euler reflection formula

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}. \tag{51}$$

(3) The Gauss multiplication formula

$$\Gamma\left[\begin{array}{c} z,z+\frac{1}{n},\cdots,z+\frac{n-1}{n} \\ - \end{array}\right]=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nz}\Gamma(nz). \tag{52}$$

We list some well-known definitions and formulas for hypergeometric series. We begin with the general items.

Definition 4. The *generalized hypergeometric series* ${}_pF_r$ is defined on the unit disk $|z|<1$ by

$${}_pF_r(a_1,\dots,a_p;b_1,\dots,b_r;z)\equiv {}_pF_r\left[\begin{array}{c} a_1,\dots,a_p \\ b_1,\dots,b_r \end{array}\middle| z\right]=\sum_{n=0}^{\infty}\frac{(a_1,\dots,a_p)_n}{n!(b_1,\dots,b_r)_n}z^n.$$

For hypergeometric series with function value equal to 1, z is not written.

Definition 5. The parametric excess (usually denoted by s) of a hypergeometric function is the sum of its lower parameters, minus the sum of its upper parameters.

These hypergeometric functions are (by analytic continuation) defined and single-valued in the complex plane $x\in\mathbb{C}$, slit along the real axis between $x=-1$ and $x=-\infty$.

Definition 6. The hypergeometric series

$${}_{r+1}F_r(a_1,\dots,a_{r+1};b_1,\dots,b_r;z)$$

is called *well-poised*, if its parameters satisfy the relations

$$1+a_1=a_2+b_1=a_3+b_2=\dots=a_{r+1}+b_r.$$

The Gauss summation formula

$${}_2F_1(a,b;c)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \mathrm{Re}(c-a-b)>0.$$

The Dixon–Schafheitlin [5, p. 82 (3.53)] summation formula for a well-poised series:

$${}_3F_2(a,b,c;1+a-b,1+a-c)=\Gamma\left[\begin{array}{c} 1+\frac{1}{2}a,1+a-b,1+a-c,1+\frac{1}{2}a-b-c \\ 1+a,1+\frac{1}{2}a-b,1+\frac{1}{2}a-c,1+a-b-c \end{array}\right],$$

provided that the series is convergent, i.e., $\mathrm{Re}\,(1+\frac{1}{2}a-b-c)>0$. This formula permits the summation of any well-poised ${}_3F_2$.

Watson–Schafheitlin summation formula [5, p. 83 (3.55)]

$${}_3\mathrm{F}_2\left(a,b,c;\frac{1}{2}(1+a+b),2c\right)=\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{2}+c,\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b,\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c\\ \frac{1}{2}+\frac{1}{2}a,\frac{1}{2}+\frac{1}{2}b,\frac{1}{2}-\frac{1}{2}a+c,\frac{1}{2}-\frac{1}{2}b+c\end{matrix}\right].$$

Whipple summation formula:

$$\begin{aligned}&{}_3\mathrm{F}_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix}\right]\\ &=\pi 2^{1-2c}\Gamma\left[\begin{matrix}d,e\\ \frac{1}{2}(a+e),\frac{1}{2}(a+d),\frac{1}{2}(1-a+e),\frac{1}{2}(1-a+d)\end{matrix}\right],\end{aligned}\tag{53}$$

where $a+b=1$ and $d+e=1+2c$.

Remark 4. The so-called Pfaff–Saalschütz summation formula applies only when $s=1$.

Theorem 9.1. *The hypergeometric series reversal formula, formula, which we have used frequently, is a corrected form of Slater* [12, p. 48]*:*

$${}_2\mathrm{F}_1\left[\begin{matrix}a,-m\\ b\end{matrix}\middle| z\right]=\frac{(a)_m}{(b)_m}(-z)^m\ {}_2\mathrm{F}_1\left[\begin{matrix}1-b-m,-m\\ 1-a-m\end{matrix}\middle| z^{-1}\right].\tag{54}$$

Formula (54) relates ${}_2\mathrm{F}_1(z)$ to ${}_2\mathrm{F}_1(z^{-1})$, and therefore relates the behavior of ${}_2\mathrm{F}_1(z)$ near $z=\infty$ to (known) behavior near $z=0$. The formula trivially extends from ${}_2\mathrm{F}_1$ to ${}_3\mathrm{F}_2$, ${}_4\mathrm{F}_3$, etc.

We recall two contiguity relations.

$$\begin{aligned}&{}_{p+2}\mathrm{F}_{p+1}\left[\begin{matrix}\alpha+1,a_0,a_1,\ldots,a_p\\ \alpha,c_1,\ldots,c_p\end{matrix}\middle| x\right]\\ &=\frac{\alpha-a_0}{\alpha}\ {}_{p+1}\mathrm{F}_p\left[\begin{matrix}a_0,a_1,\ldots,a_p\\ c_1,\ldots,c_p\end{matrix}\middle| x\right]+\frac{a_0}{\alpha}\ {}_{p+1}\mathrm{F}_p\left[\begin{matrix}a_0+1,a_1,\ldots,a_p\\ c_1,\ldots,c_p\end{matrix}\middle| x\right].\end{aligned}\tag{55}$$

$$\begin{aligned}&{}_{p+2}\mathrm{F}_{p+1}\left[\begin{matrix}\alpha+1,a_0,a_1,\ldots,a_p\\ \alpha,c_1,\ldots,c_p\end{matrix}\middle| x\right]\\ &=\ {}_{p+1}\mathrm{F}_p\left[\begin{matrix}a_0,a_1,\ldots,a_p\\ c_1,\ldots,c_p\end{matrix}\middle| x\right]\\ &\quad+\frac{\prod_{i=0}^{p}a_i}{\alpha\prod_{i=1}^{p}c_i}\ {}_{p+1}\mathrm{F}_p\left[\begin{matrix}a_0+1,a_1+1,\ldots,a_p+1\\ c_1+1,\ldots,c_p+1\end{matrix}\middle| x\right].\end{aligned}\tag{56}$$

10. Acknowledgements.

Firstly, I want to thank Sergei Silvestrov for guiding me into the realm of special functions by giving me many of the problems, which are solved in my first two books.

I thank Arne Meurman for good advice on hypergeometric series and much more. Special thanks to Leif Karlsson for linguistic improvements of the text. And special thanks to the referee for his generous comments, which have improved the text substantially.

I thank Henrik Karlsson, for letting me use Per Karlsson's big library.

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Received June 29, 2023