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Statuses and double branch weights of quadrangular outerplanar graphs

Abstract. In this paper we study some distance properties of outerplanar graphs with the Hamiltonian cycle whose all bounded faces are cycles isomorphic to the cycle C_4. We call this family of graphs quadrangular outerplanar graphs. We give the lower and upper bound on the double branch weight and the status for this graphs. At the end of this paper we show some relations between median and double centroid in quadrangular outerplanar graphs.

1. Introduction. Let $G = (V(G), E(G))$ be a simple, finite and connected graph without loops. Let $w : V(G) \cup E(G) \to R^+$ be a weight function. The pair (G, w) is called a *weighted graph*. For a path P in G, the *weight length* of P, denoted by $l_w(P)$, is defined by

$$l_w(P) = \sum_{e \in E(P)} w(e).$$

For any two vertices $x, y \in V(G)$, the *weight distance* between x and y, denoted by $d_w(x, y)$, is defined as

$$d_w(x, y) = \min\{l_w(P)\},$$

where the minimum is taken over all paths P joining x and y.

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For any vertex x of G, the *status* of x, denoted by $s(x)$, is defined by

$$s(x) = \sum_{y \in V(G)} w(y) d_w(y, x).$$

The *status* of G, denoted by $S(G)$, is defined as

$$S(G) = \min\{s(x) : x \in V(G)\}.$$

The *median* of G, denoted by $M_1(G)$, is the set of vertices in G with the smallest status, i.e.,

$$M_1(G) = \{t \in V(G) : s(t) = S(G)\}.$$

The *second median* of G, denoted by $M_2(G)$, is the set of vertices in G with the second smallest status, i.e.,

$$M_2(G) = \{t \in V(G) - M_1(G) : s(t) \leq s(x) \text{ for all } x \in V(G) - M_1(G)\}.$$

The *weight* of the graph G, denoted by $w(G)$, is defined by

$$w(G) = \sum_{x \in V(G)} w(x).$$

Let $S \subseteq V(G)$. By $G - S$ we denote the graph obtained from G by deleting all vertices of S with the edges incident to S. For notation and terminology not defined in this paper, the reader is referred to [1].

Authors of the papers [2]–[14] studied medians, centroids and other special sets of vertices of graphs. Most of the results they obtained for trees, unicyclic graphs and some other classes of graphs. The subject is related to network location problems. First we recall some previous results relevant to trees. Let (T, w) be a weighted tree. For the tree T we have the following definitions.

For any vertex x of the weighted tree T, the *branch weight* of x, denoted by $bw(x)$, is the maximum weight of any component of $T - \{x\}$. The *centroid* of T, denoted by $C_1(T)$, is the set of vertices in T with the smallest branch weight. The *second centroid* of T, denoted by $C_2(T)$, is the set of vertices in T with the second smallest branch weight.

Zelinka in [14] proved the following interesting results.

Theorem 1.1 (Zelinka [14]). *A tree has in median either exactly one vertex, or exactly two vertices joined by an edge.*

Theorem 1.2 (Zelinka [14]). *Any tree with constant weight function has its median equal to its centroid.*

Kariv and Hakimi studied weighted trees and discovered the following relation between centroid and branch weight.

Theorem 1.3 (Kariv, Hakimi [5]). *Let (T, w) be a weighted tree and x be a vertex in T. Then x is in centroid of T if and only if*

$$bw(x) \leq \frac{1}{2}w(T).$$

For unweighted trees the above result was also proved by Szamkołowicz [12]. Recently Lin and Shang described the property of the second median and the second centroid in weighted trees.

Theorem 1.4 (Lin, Shang [7]). *Let (T, w) be a weighted tree with $w(e) = 1$ for each $e \in E(T)$. Then*

$$M_2(T) = C_2(T).$$

Now we present some interesting result of Lin et al. [8]. They proved the lower and upper bound on the status of a connected unweighted graph G in terms of some spanning trees of G. So we need the following notions and notations.

Let T be a unweighted rooted tree with the root z. In this case $w(x) = w(e) = 1$ for each vertex $x \in V(T)$ and each edge $e \in E(T)$. Then the *height* of T is defined as

$$height(T) = \max\{d(z, x) : x \in V(T)\},$$

where $d(z, x)$ is a distance between z and x.

Let $height(T) \geq 2$ and suppose that l is an integer greater or equal to 2. The tree T is called a *balanced l-tree* if $deg_T(x) = l$ for each $x \in V(T)$ with $d(z, x) \leq height(T) - 2$. A balanced l-tree of order n is denoted by $B_{n,l}$.

For integers $n > l \geq 2$, the *l-grass* $G_{n,l}$ is the graph with

$$V(G_{n,l}) = \{x_1, x_2, \ldots, x_n\}$$

and

$$E(G_{n,l}) = \{x_i x_{i+1} : i = 1, 2, \ldots, n-l\} \cup \{x_{n-l+1} x_j : j = n-l+2, n-l+3, \ldots, n\}.$$

Note that $G_{n,l}$ has n vertices and has a vertex with degree l (more specific $deg(x_{n-l+1}) = l$).

Theorem 1.5 (Lin, et al. [8]). *Let G be a connected graph of order n and with maximum degree $l \geq 2$. Then*

$$S(B_{n,l}) \leq S(G) \leq S(G_{n,l}).$$

Furthermore, the lower bound is attained if and only if G contains some balanced l-tree $B_{n,l}$ as a spanning subgraph, and if the upper bound is attained, then G contains the l-grass $G_{n,l}$ as a spanning subgraph.

We extend some of the results to a subclass of weighted outerplanar graphs. For this purpose we define some additional notions and notations. Let $G = (V(G), E(G))$ be a weighted outerplanar graph.

For any two vertices x and y of G, where $\{x, y\} \in E(G)$, we define the *edge double branch weight* of an edge $\{x, y\}$, denoted by $dbw(\{x, y\})$, as the maximum weight of any component of $G - \{x, y\}$. The *double branch weight* of G, denoted by $DBW(G)$, is defined as

$$DBW(G) = \min\{dbw(\{x, y\}) : x, y \in V(G), \{x, y\} \in E(G)\}.$$

The *vertex double branch weight* of a vertex $x \in V(G)$ we define as follows

$$dbw^*(x) = \min\{dbw(\{x, y\}) : y \in V(G), \{x, y\} \in E(G)\}.$$

The *double centroid* of G, denoted by $DC_1(G)$, is the set of vertices in G with the smallest vertex double branch weight. The *second double centroid* of G, denoted by $DC_2(G)$, is the set of vertices in G with the second smallest vertex double branch weight. The *minimal separator* of G, is the subgraph of G, which is composed of two vertices x, y and the edge $\{x, y\}$ such that $dbw(\{x, y\}) = DBW(G)$, where $x, y \in DC_1(G)$. The *union of minimal separators* of G, denoted by $MS(G)$, is the subgraph of G, which is composed of vertices from $DC_1(G)$ and edges $\{x, y\}$ such that $dbw(\{x, y\}) = DBW(G)$, where $x, y \in DC_1(G)$.

Let us consider the following example to understand the above definitions.

Example 1.6. Consider the weighted outerplanar graph G with 12 vertices, $w(e) = 1$ for each $e \in E(G)$ and $w(x) = 1$ for each $x \in V(G)$ presented in Figure 1.

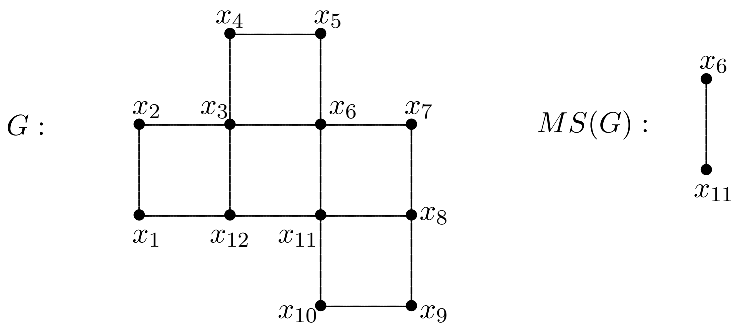

FIGURE 1. The graph G with $MS(G)$.

We see that

$$M_1(G) = \{x_6, x_{11}\} \text{ and } M_2(G) = \{x_3, x_{12}\},$$

$$DC_1(G) = \{x_6, x_{11}\} \text{ and } DC_2(G) = \{x_3, x_8, x_{12}\}.$$

Additionally we have

$$DBW(G) = 6$$

and $MS(G)$ is a subgraph presented in Figure 1.

2. Main results for the family of quadrangular outerplanar graphs. We study some distance properties of weighted outerplanar graphs with the Hamiltonian cycle whose all bounded faces are cycles isomorphic to the cycle C_4. We call these graphs quadrangular outerplanar graphs.

Let (G, w) be a weighted quadrangular outerplanar graph of order $n > 3$. Notice that n is even for these graphs. We begin this section with a theorem which states that the set of vertices with the smallest vertex double branch weight is composed of two, three or four vertices. Next we prove the lower and upper bound of double branch weight $DBW(G)$ and give an algorithm for constructing quadrangular outerplanar graphs with some interesting properties. The last theorem concerns bounds for status $S(G)$. At the end of this section we define a distance between median and double centroid and try to find some relations between them in quadrangular outeplanar graphs.

Let us start from a theorem that the cardinality of double centroid for quadrangular outerplanar graphs belongs to the set $\{2, 3, 4\}$.

Theorem 2.1. *Let (G, w) be a weighted quadrangular outerplanar graph with n vertices and let $w(x) = c$ for each $x \in V(G)$, where c is a constant positive real number. Then the double centroid of graph G consists of two, three or four vertices which lie in the same cycle C_4. Moreover, the union of minimal separators of graph G has one of five possible structures:*

(i) *a path P_2,*

(ii) *two disjoint copies of a path P_2 (both paths lie in the same quadrangle as proper subgraph of C_4),*

(iii) *a path P_3 (edges lie in the same quadrangle as a proper subgraph of C_4),*

(iv) *a path P_4 (edges lie in the same quadrangle as a proper subgraph of C_4),*

(v) *a cycle C_4.*

Proof. Let (G, w) be a weighted quadrangular outerplanar graph with n vertices, $w(x) = c$ for each $x \in V(G)$, where c is a constant positive real number.

By definition we get that the double centroid of any outerplanar graph consists of at least two vertices which are adjacent. It is easy to construct quadrangular outerplanar graph for which the union of minimal separators has the structure P_2, $2P_2$, P_3, P_4 or C_4. The five respective constructions are presented in Figures 2–6, where graphs G_1, G_2 are components constructed by deleting the cut vertex set $\{x, y\}$ from the graph G.

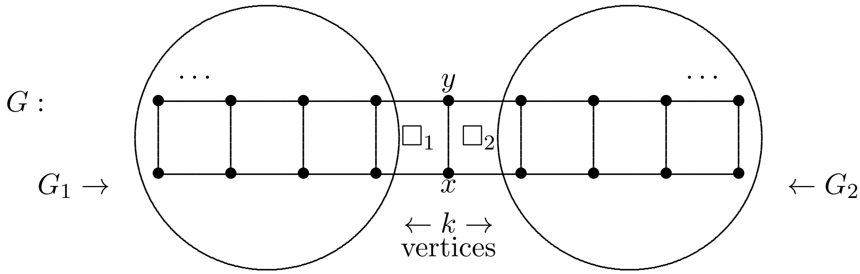

FIGURE 2. Quadrangular outerplanar graph of order $n \geq 6$ with the path $P_2 = xy$ as the union of minimal separators; $k = \frac{n-2}{2}$.

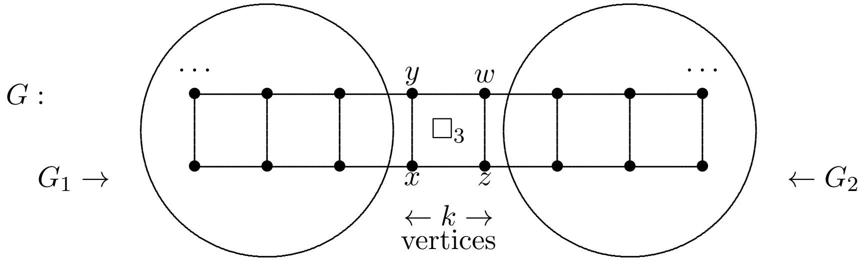

FIGURE 3. Quadrangular outerplanar graph of order $n \geq 8$ with two disjoint paths P_2 (xy and zw) as the union of minimal separators; $k = \frac{n-4}{2}$

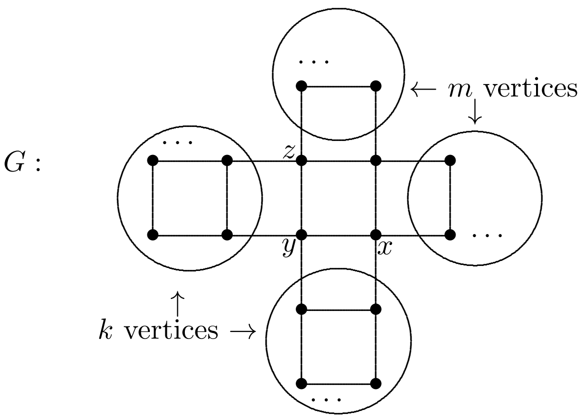

FIGURE 4. Quadrangular outerplanar graph of order $n \geq 8$ with the path $P_3 = xyz$ as the union of minimal separators; k, m – even natural numbers such that $k > m$ and $n = 2(k + m) + 4$.

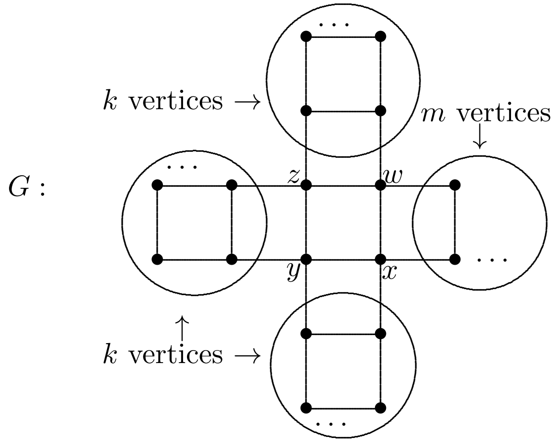

FIGURE 5. Quadrangular outerplanar graph of order $n \geq 10$ with the path $P_4 = xyzw$ as the union of minimal separators; k, m – even natural numbers such that $k > m$ and $n = 3k + m + 4$.

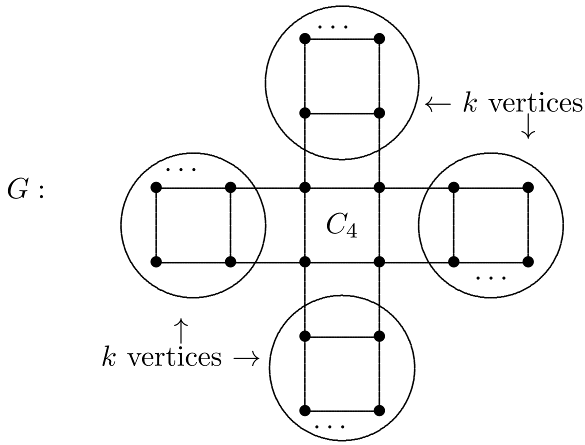

FIGURE 6. Balanced quadrangular outerplanar graph of order n with the cycle C_4 as the union of minimal separators. This structure can be obtained only for $n \equiv 4 \bmod 8$; $k = \frac{n-4}{4}$.

Now we prove that the minimal separators of quadrangular outerplanar graph can not lie in different cycles C_4. It implies the fact that the double centroid can not be composed of more than four vertices. Observe that for $n > 4$ minimal separator can not lie on the Hamiltonian cycle. Consider two cases.

Case 1. First we show that the minimal separators can not lie in different cycles C_4 and be separable. Assume that it is not true. Let $\{x, y\}$ and

$\{z, w\}$, where $x, y, z, w \in V(G)$, be two minimal separators of graph G such that

$$\exists_{p,q \in \{x,y,z,w\}}\ d(p, q) > 2bc,$$

where $d(p, q)$ is the weight distance between vertices p and q.

Now let G_1, G_2 be components of $G - \{x, y\}$ and G_3, G_4 be components of $G - \{z, w\}$. Observe that by definition

$$\{x, y\} \notin G_1 \text{ and } \{z, w\} \notin G_4.$$

Assume, without loss of generality, that

$$\{x, y\} \in G_3 \text{ and } \{z, w\} \in G_2.$$

Note that

$$G_1 \subset G_3 \text{ and } G_4 \subset G_2.$$

Thus

$$|G_3| \geq |G_1| + 4 \text{ and } |G_2| \geq |G_4| + 4.$$

Since $\{x, y\}$ and $\{z, w\}$ are minimal separators, then we have two possibilities

$$|G_1| = |G_4| \text{ or } |G_2| = |G_3|.$$

Since both separators are disjoint, then there exists an edge $\{s, t\}$, where $s, t \in V(G)$ such that G_5 and G_6 are components of $G - \{s, t\}$ and without loss of generality x or $y \in G_5$ and z or $w \in G_6$. So the following inequalities are true

$$|G_1| < |G_5| < |G_3|, \tag{2.1}$$

$$|G_4| < |G_6| < |G_2|. \tag{2.2}$$

We have to consider two subcases.

Case 1.1. $DBW(G) = c|G_1| = c|G_4|$. Hence $|G_1| > |G_2|$ and $|G_4| > |G_3|$. Then we have

- if $dbw(\{s, t\}) = c|G_5|$, then by (2.1) we get $|G_5| \geq |G_1| = |G_4| > |G_3| > |G_5|$, a contradiction,
- if $dbw(\{s, t\}) = c|G_6|$, then by (2.2) we get $|G_6| \geq |G_1| > |G_2| > |G_6|$, a contradiction.

Case 1.2. $DBW(G) = c|G_2| = c|G_3|$. Hence $|G_2| > |G_1|$ and $|G_3| > |G_4|$. Then we have

- if $dbw(\{s, t\}) = c|G_5|$, then by (2.1) we get $|G_5| \geq |G_2| = |G_3| > |G_5|$, a contradiction,
- if $dbw(\{s, t\}) = c|G_6|$, then by (2.2) we get $|G_6| \geq |G_2| > |G_6|$, a contradiction.

We obtain a contradiction. Thus our assumption is false.

Case 2. Now we show that the minimal separators which have a common vertex can not lie in the different cycles C_4. Assume that it is not true. Let $\{x, y\}$ and $\{x, z\}$, where $x, y, z \in V(G)$ be two minimal separators of graph G such that

$$\{x, y\} \cap \{x, z\} = \{x\}.$$

As previous, let G_1, G_2 be components of $G - \{x, y\}$ and G_3, G_4 be components of $G - \{x, z\}$. Observe that by definition

$$y \notin G_1 \text{ and } z \notin G_4.$$

Assume, without loss of generality, that

$$y \in G_3 \text{ and } z \in G_2.$$

Note that

$$G_1 \subset G_3 \text{ and } G_4 \subset G_2.$$

Thus

$$|G_3| \geq |G_1| + 4 \text{ and } |G_2| \geq |G_4| + 4.$$

Since $\{x, y\}$ and $\{x, z\}$ are minimal separators, then we have two possibilities

$$|G_1| = |G_4| \text{ or } |G_2| = |G_3|.$$

Since both separators lie in different cycles C_3, then there exists an edge $\{x, v\}$, where $v \in V(G)$ such that G_5 and G_6 are components of $G - \{x, v\}$ and without loss of generality $y \in G_5$ and $z \in G_6$. So the following inequalities are true

$$|G_1| < |G_5| < |G_3|, \tag{2.3}$$

$$|G_4| < |G_6| < |G_2|. \tag{2.4}$$

We have to consider two subcases, the same as in Case 1.

Case 2.1. $DBW(G) = c|G_1| = c|G_4|$. Hence $|G_1| > |G_2|$ and $|G_4| > |G_3|$. Then we have

- if $dbw(\{x, v\}) = c|G_5|$, then by (2.3) we get $|G_5| \geq |G_1| = |G_4| > |G_3| > |G_5|$, a contradiction,
- if $dbw(\{x, v\}) = c|G_6|$, then by (2.4) we get $|G_6| \geq |G_1| > |G_2| > |G_6|$, a contradiction.

Case 2.2. $DBW(G) = c|G_2| = c|G_3|$. Hence $|G_2| > |G_1|$ and $|G_3| > |G_4|$. Then we have

- if $dbw(\{x, v\}) = c|G_5|$, then by (2.3) we get $|G_5| \geq |G_2| = |G_3| > |G_5|$, a contradiction,
- if $dbw(\{x, v\}) = c|G_6|$, then by (2.4) we get $|G_6| \geq |G_2| > |G_6|$, a contradiction.

We obtain a contradiction. Thus our assumption is false. All minimal separators of a quadrangular outerplanar graph have to lie in the same cycle C_4. $\square$

Now we show and prove the lower and upper bound for the value of double branch weight of weighted quadrangular outerplanar graph G. We also give an exemplary algorithm which allows us to construct weighted quadrangular outerplanar graphs with some properties of $DBW(G)$ and the union of minimal separators $MS(G)$.

We start from a definition of an auxiliary function. Let n be a number of vertices in a quadrangular outerplanar graph and let $\mathbb{N}_2$ be a set of natural even numbers. For n we define a function

$$\phi : \mathbb{N}_2 \to \{0, 1\},$$

by the formula

$$\phi(n) = \begin{cases} 1, & \text{when } n \equiv 0 \bmod 8 \vee n \equiv 6 \bmod 8, \\ 0, & \text{when } n \equiv 2 \bmod 8 \vee n \equiv 4 \bmod 8. \end{cases}$$

Theorem 2.2. *Let (G, w) be a weighted quadrangular outerplanar graph with n vertices, $w(x) = c$ for each $x \in V(G)$, where c is a constant real positive number. Then we have*

$$2c \left\lfloor \frac{n}{4} \right\rfloor \leq DBW(G) \leq c\left(\left\lfloor \frac{3n-4}{4} \right\rfloor - \phi(n) \right).$$

Moreover, let

$$L = 2c \left\lfloor \frac{n}{4} \right\rfloor,$$

$$U = c\left(\left\lfloor \frac{3n-4}{4} \right\rfloor - \phi(n) \right),$$

$$\text{and } z \in \{L, L + 2c, L + 4c, \ldots, U - 2c, U\}.$$

Then for all values of z there exists a quadrangular outerplanar graph G such that $DBW(G) = z$.

Proof. We will construct graphs which hold the above inequalities. Let $n = 4$ and C_4 be the smallest weighted quadrangular outerplanar graph. In this case we have

$$2c \left\lfloor \frac{n}{4} \right\rfloor = 2c \left\lfloor \frac{4}{4} \right\rfloor = 2c,$$

$$c\left(\left\lfloor \frac{3n-4}{4} \right\rfloor - \phi(n) \right) = c\left(\left\lfloor \frac{3 \cdot 4 - 4}{4} \right\rfloor - \phi(4) \right) = 2c.$$

We know that

$$DBW(C_4) = 2c.$$

Thus both inequalities hold. Let $n > 4$. By addition of an even number of vertices to a graph C_4, we obtain a new weighted quadrangular outerplanar graph.

Let x and y be vertices in G such that $\{x, y\} \in E(G)$ and $dbw(\{x, y\}) \leq dbw(\{z, w\})$ for each $\{z, w\} \in E(G)$ and let G_1, G_2 be components in $G - \{x, y\}$. Then we have

$$w(G) = w(G_1) + w(G_2) + w(x) + w(y).$$

By $w(x) = w(y) = c$ we have

$$w(G) = w(G_1) + w(G_2) + 2c. \tag{2.5}$$

Case 1. Consider the situation when subsequent vertices are added to one or two edges in C_4 (see Figures 2 and 3). In this case the value of the double branch weight of graph G is minimal.

We distinguish two subcases.

Case 1.1. If n is divisible by 4, then $w(G_1) = w(G_2) + 2c$ and by (2.5) we have $w(G_1) = \frac{1}{2}w(G)$.

Case 1.2. If n is not divisible by 4, then $w(G_1) = w(G_2)$ and by (2.5) we have $w(G_1) = \frac{1}{2}w(G) - c$. Thus $DBW(G) = dbw(\{x, y\}) \geq w(G_1) = 2c \left\lfloor \frac{n}{4} \right\rfloor$.

Case 2. Now consider the situation when subsequent vertices are added to all edges in C_4 (in balanced way if it is possible; see Figure 7). In this case the value of the double branch weight of the graph G is maximal.

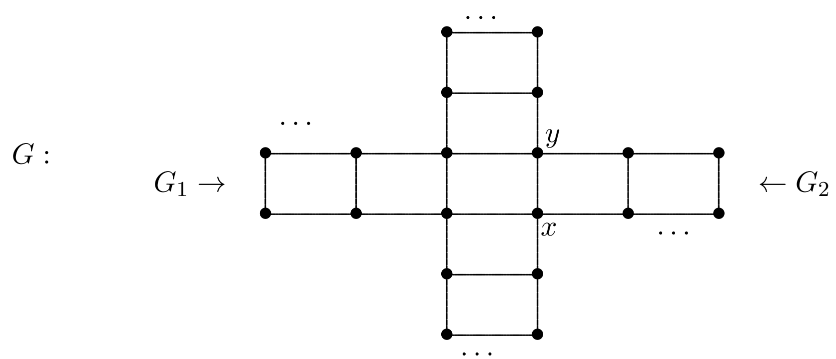

FIGURE 7. Balanced quadrangular outerplanar graph with the path $P_2 = xy$ as a minimal separator.

We distinguish four subcases.

Case 2.1. If $n \equiv 0 \mod 8$, then $w(G_1) = 3w(G_2) - 2c$ and by (2.5) we have $w(G_1) = \frac{3}{4}w(G) - 2c$.

Case 2.2. If $n \equiv 2 \mod 8$, then $w(G_1) = 3w(G_2)$ and by (2.5) we have $w(G_1) = \frac{3}{4}w(G) - \frac{3}{2}c$.

Case 2.3. If $n \equiv 4 \mod 8$, then $w(G_1) = 3w(G_2) + 2c$ and by (2.5) we have

$w(G_1) = \frac{3}{4}w(G) - c$.
Case 2.4. If $n \equiv 6 \bmod 8$, then $w(G_1) = 3w(G_2) - 4c$ and by (2.5) we have $w(G_1) = \frac{3}{4}w(G) - \frac{5}{2}c$. Thus $DBW(G) = dbw(\{x, y\}) \leq w(G_1) = c\left(\left\lfloor \frac{3n-4}{4} \right\rfloor - \phi(n)\right)$. This completes the proof of the inequalities.

Now we prove the second part of Theorem 2.2. Let $n > 3$ be a positive even integer. An algorithm for constructing quadrangular outerplanar graphs with value $DBW(G) = z$, where $z \in \{L, L+2c, L+4c, \ldots, U-2c, U\}$, is given below. Additionally graphic structure of the union of minimal separators must be saved. Let as consider four cases.

Case A. Let $n \equiv 6 \bmod 8$.

Step A.1. Value $DBW(G) = L$ is achieved by arrangement of all vertices as in Figure 2.

The union of minimal separators of G has structure P_2. In the graph G we use two quadrangles, $\square_1$ and $\square_2$, which have a common edge (see Figure 2).

Step A.2. To achieve the value $DBW(G) = L+2c$ we delete four furthest vertices from horizontal part of the graph, delete the edges incident to them and attach these vertices either on the quadrangle $\square_1$ or on the quadrangle $\square_2$ as in Figure 8(a).

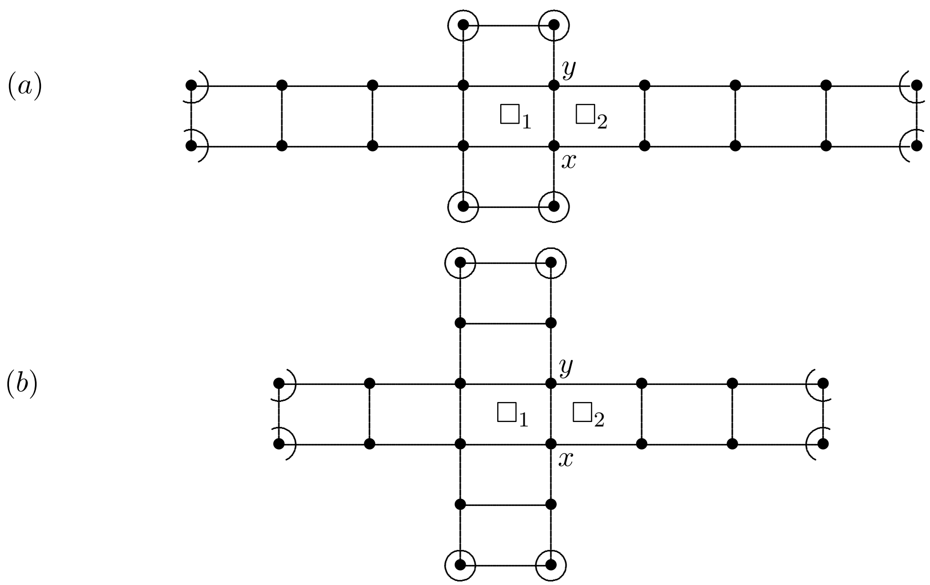

FIGURE 8. Quadrangular outerplanar graph of order n with the path $P_2 = xy$ as the union of minimal separators. Cut the half-circled vertices and edges incident to them, and put them into circled place up and down the quadrangle $\square_1$.

Step A.3. To achieve next values $L+4c, \ldots, U-c, U$ we do the same as in the step A.2. Precisely, we delete four furthest vertices from the horizontal part, delete the edges connecting them with the graph and tack these vertices on the vertical part of the graph G (up and down the quadrangle chosen before, e.i. $\square_1$) as in Figure 8(b).

Case B. Let $n \equiv 2 \bmod 8$.

Step B.1. To achieve values $DBW(G) = L, L+2c, \ldots, U-2c$ we iteratively apply the procedure as in case A.

Step B.2. To achieve value $DBW(G) = U$ we equinumerously add vertices to all edges in the quadrangle C_4 (the structure of the union of minimal separators in this step is changed on the path P_4 as in Figure 5).

Case C. Let $n \equiv 0 \bmod 8$.

Step C.1. Value $DBW(G) = L$ is achieved by arrangement of all vertices as in Figure 3. The union of minimal separators of G has the structure $2P_2$. In the graph G we have the quadrangle $\square_3$ (see Figure 3).

Step C.2. To achieve the value $DBW(G) = L+2c$ we start from the configuration presented in step C.1 and move four furthest vertices from the horizontal part and tack them on the quadrangle $\square_3$ as in Case A.2.

Case C.3. To achieve next values $L+4c, \ldots, U-2c, U$ we iteratively do the same as in the step C.2. Precisely, we move four furthest vertices and tack them on the place up and down the quadrangle $\square_3$ as in Case A.2.

Case D. Let $n \equiv 4 \bmod 8$.

Step D.1. To achieve values $DBW(G) = L, L+2c, \ldots, U-2c$ we do the same as in the case C.

Step D.2. To achieve value $DBW(G) = U$ we do the same as in case B in the step B.2 (the structure of the union of minimal separators in this step is changed on the cycle C_4 as in Figure 6). $\square$

Remark 2.3. The algorithm presented in the proof of Theorem 2.2 gives one of many possible ways to construct a weighted quadrangular outerplanar graph G with the property $DBW(G) = z$, where z belongs to the set presented in the Theorem 2.2.

In the next part of the main section we give the lower and upper bound for the value of status of quadrangular outerplanar graph G. The bounds presented in Theorem 1.5 depend on the order and maximum degree of the graph. In our theorem these bounds depend on the order of the graph and a weight function.

Theorem 2.4. *Let (G, w) be a weighted quadrangular outerplanar graph with n vertices and let $w(e) = b$ for each edge $e \in E(G)$, $w(x) = c$ for each $x \in V(G)$, where b, c are constant positive real numbers. Then*

$$b \cdot c \cdot \left(\frac{3}{2}n - 2\right) \leq S(G) \leq b \cdot c \cdot K,$$

where

$$K = \begin{cases} \frac{n}{4}\left(\frac{n}{2}+2\right), \ when\ n \equiv 0 \ mod\ 4, \\ \lfloor\frac{n}{4}\rfloor\left(\frac{n}{2}+3\right)+1, \ when\ n \equiv 2 \ mod\ 4. \end{cases}$$

Proof. Let (G, w) be a weighted quadrangular outerplanar graph with n vertices and let $w(e) = b$ for each edge $e \in E(G)$, $w(x) = c$ for each $x \in V(G)$, $b, c \in \mathbb{R}^+$. By definition we have $S(G) = s(x)$, where $x \in M_1(G)$.

First we show the lower bound. Observe that the status S(G) has a minimum value if vertex $x \in M_1(G)$ has weight distances $d_w(x, v) = b$ for $\frac{n}{2}$ vertices and $d_w(x, v) = 2b$ for $\frac{n}{2} - 1$ vertices, where $v \in V(G)$ and $x \neq v$. The quadrangular outerplanar which satisfies the condition is presented in Figure 9.

G :

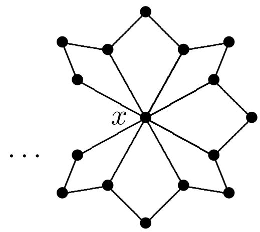

FIGURE 9. Quadrangular outerplanar graph G of order n with minimal value of the status $S(G)$ and $M_1(G) = \{x\}$.

Since $s(x) = \min\{s(y) : y \in V(G)\} = b \cdot c \cdot \left(\frac{3}{2}n - 2\right)$, then $S(G) \geq s(x) = b \cdot c \cdot \left(\frac{3}{2}n - 2\right)$.

Now we prove the upper bound. Let us start from the cycle C_4 ($G \cong C_4$) with vertices x, y, z, w. It is known that $M_1(C_4) = V(C_4)$. For C_4 we have $S(C_4) = 4bc$.

By addition of new vertices to the cycle C_4, we obtain a new quadrangular outerplanar graph G. Observe that the value of status $S(G)$ is maximal if new vertices are added symmetrically to two edges of C_4 (in horizontal way). If $x \in M_1(C_4)$, then by this construction we have $x \in M_1(G)$. Moreover, x has maximal weight distances $d_w(x, v)$, where $v \in V(G)$. Figure 10 presents this situation. We can split vertices from $V(G) - \{x\}$ on two sets:

- $V_u(G)$ - vertices which are not on the same horizontal line as vertex x (upper level),
- $V_d(G)$ - vertices which are on the same horizontal line as vertex x (lower level).

We have

$$s(x) = s_u(x) + s_d(x), \tag{2.6}$$

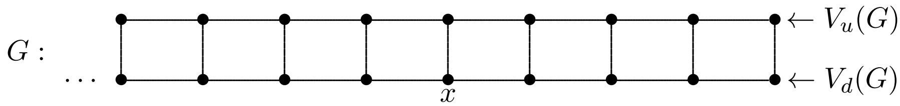

FIGURE 10. Subdivision of the vertex set of quadrangular outerplanar graph G of order n at lower $V_d(G)$ and upper $V_u(G)$ horizontal lines.

where

$$s_u(x) = \sum_{y \in V_u(G)} w(y) d_w(y, x),$$

$$s_d(x) = \sum_{y \in V_d(G)} w(y) d_w(y, x).$$

We have to distinguish two cases.

Case 1. Let $n \equiv 0 \bmod 4$. Then $|V_u(G)| = \frac{n}{2}$ and $|V_d(G)| = \frac{n}{2} - 1$. Thus

$$s_u(x) = bc\left(1 + 2 + \cdots + \frac{n}{4} + 2 + 3 + \cdots + \left(\frac{n}{4} + 1\right)\right) = bc\frac{n}{4}\left(\frac{n}{4} + 2\right),$$

$$s_d(x) = 2bc\left(1 + 2 + \cdots + \left(\frac{n}{4} - 1\right)\right) + bc\frac{n}{4} = bc\left(\frac{n}{4}\right)^2.$$

Then by (2.6) we obtain

$$S(G) \leq s(x) = bc\left(\frac{n}{4}\left(\frac{n}{2} + 2\right)\right).$$

Case 2. Let $n \equiv 2 \bmod 4$. Then $|V_u(G)| = \frac{n}{2}$ and $|V_d(G)| = \frac{n}{2} - 1$. Thus

$$s_u(x) = 2bc\left(2 + 3 + \cdots + \left(\left\lfloor\frac{n}{4}\right\rfloor + 1\right)\right) + bc = bc\left(\left\lfloor\frac{n}{4}\right\rfloor\left(\left\lfloor\frac{n}{4}\right\rfloor + 3\right) + 1\right),$$

$$s_d(x) = 2bc\left(1 + 2 + \cdots + \left\lfloor\frac{n}{4}\right\rfloor\right) = bc\left\lfloor\frac{n}{4}\right\rfloor\left(\left\lfloor\frac{n}{4}\right\rfloor + 1\right).$$

Then by (2.6) we obtain

$$S(G) \leq s(x) = bc\left(\left\lfloor\frac{n}{4}\right\rfloor\left(\frac{n}{2} + 3\right) + 1\right).$$

This completes the proof of inequalities. □

Now let us define a distance between median and double centroid. We also give some examples of constructions where this distance is equal to 0 or 1.

Let (G, w) be a weighted outerplanar graph with n vertices, $w(e) = b$ for each $e \in E(G)$, $w(x) = c$ for each $x \in V(G)$, where b, c are constant positive real numbers. Let $M_1(G)$ be a median of the graph G and $DC_1(G)$ be a double centroid of the graph G.

Definition 2.5. The *distance between median and double centroid* of the graph G, denoted by $d^*(G)$, is defined by

$$d^*(G) = \begin{cases} 0, & M_1(G) \cap DC_1(G) \neq \emptyset, \\ d, & M_1(G) \cap DC_1(G) = \emptyset, \end{cases}$$

where $d = \min\{d_w(x, y)\}$, and the minimum is taken over all weight distances between $x \in M_1(G)$ and $y \in DC_1(G)$.

Example 2.6. Let us consider a weighted quadrangular outerplanar graph (G, w) with 12 vertices presented in Figure 1. Let $w(x) = w(e) = 1$ for each $x \in V(G)$ and for each $e \in E(G)$. We have $M_1(G) = \{x_6, x_{11}\} = DC_1(G)$. Thus $M_1(G) \cap DC_1(G) \neq \emptyset$ and $d^*(G) = 0$.

Example 2.7. Let us consider a weighted quadrangular outerplanar graph (G, w) with $n \geq 32$ vertices, $n \equiv 0 \mod 8$, presented in Figure 11, where $k = \frac{n}{4}$. Let $w(x) = c$ for each $x \in V(G)$, $w(e) = b$ for each $e \in E(G)$, $b, c \in \mathbb{R}^+$. Notice that $M_1(G) = \{x\}$ and $DC_1(G) = \{y, z, w\}$. Hence $M_1(G) \cap DC_1(G) = \emptyset$ and $d^*(G) = 1$.

Thus we have a construction where the median and the double centroid are two disjoint sets.

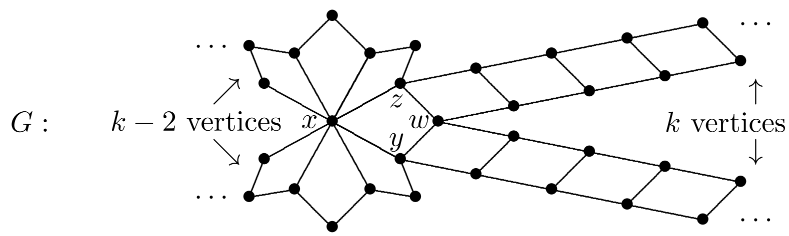

FIGURE 11. Quadrangular outerplanar graph G of order n, where $n \equiv 0 \mod 8$ and $k = \frac{n}{4}$. $M_1(G) \cap DC_1(G) = \emptyset$.

By above examples we have the following corollary.

Corollary 2.8. *Let b, c be constant positive real numbers. There exists a weighted quadrangular outerplanar graph (G, w) with $w(e) = b$ for each $e \in E(G)$ and $w(x) = c$ for each $x \in V(G)$, such that $d^*(G) \in \{0, 1\}$.*

We conclude with the following problem. Does there exist a weighted quadrangular outerplanar graph (G, w) with constant weight functions on vertices and edges, for which $d^*(G) > 1$?

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