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Properties of functions concerned with Carathéodory functions

Abstract. Let $\mathcal{P}_n$ denote the class of analytic functions $p(z)$ of the form $p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots$ in the open unit disc $\mathbb{U}$. Applying the result by S. S. Miller and P. T. Mocanu (J. Math. Anal. Appl. **65** (1978), 289–305), some interesting properties for $p(z)$ concerned with Carathéodory functions are discussed. Further, some corollaries of the results concerned with the result due to M. Obradović and S. Owa (Math. Nachr. **140** (1989), 97–102) are shown.

1. Introduction. Let $\mathcal{A}_n$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n = 1, 2, 3, \dots) \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. If a function $f(z) \in \mathcal{A}_n$ satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U}), \tag{1.2}$$

then $f(z)$ is said to be starlike with respect to the origin in $\mathbb{U}$. We denote by $\mathcal{S}_n^*$ the subclass of $\mathcal{A}_n$ consisting of functions $f(z)$ which are starlike with respect to the origin in $\mathbb{U}$. From the definition of the class $\mathcal{S}_n^*$, we see that

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if $f(z) \in \mathcal{A}_n$ satisfies

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}), \tag{1.3}$$

then $f(z) \in \mathcal{S}_n^*$. We denote by $\mathcal{T}_n^*$ the subclass of $\mathcal{S}_n^*$ consisting of $f(z)$ satisfying (1.3).

Obradović and Owa [5] have shown the following result:

Theorem A. *If $f(z) \in \mathcal{A}_1$ satisfies $f(z)f'(z) \neq 0$ for $0 < |z| < 1$ and*

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{5}{4} \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathbb{U}), \tag{1.4}$$

then $f(z) \in \mathcal{T}_1^*$.

In order to discuss our results, we have to recall here the following lemma due to Miller and Mocanu [3] (also due to Jack [2]):

Lemma 1.1. *Let*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a_n \neq 0)$$

be analytic in $\mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ on the circle $|z| = r < 1$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)|, \tag{1.5}$$

then we can write

$$z_0 w'(z_0) = m w(z_0), \tag{1.6}$$

where m is real and $m \geq n$.

Example 1.1. We consider the function $w(z)$ given by

$$w(z) = z^n + \frac{e^{i\theta}}{n+1} z^{n+1} \quad (n = 1, 2, 3, \dots). \tag{1.7}$$

Then, it follows that

$$\max_{|z| \leq |z_0|} |w(z)| = \max_{|z| \leq |z_0|} |z|^n \left| 1 + \frac{e^{i\theta} z}{n+1} \right| \leq r^n \left(1 + \frac{r}{n+1} \right) \tag{1.8}$$

for $z_0 = re^{-i\theta} \in \mathbb{U}$. This shows that $|w(z)|$ attains its maximum value at a point $z_0 \in \mathbb{U}$ on the circle $|z| = r$. For such a point $z_0 = re^{-i\theta}$, we have that

$$\frac{z_0 w'(z_0)}{w(z_0)} = \frac{z_0^n (n + e^{i\theta} z_0)}{z_0^n \left(1 + \frac{e^{i\theta} z_0}{n+1} \right)} = \frac{(n+1)(n+r)}{n+1+r} = m \geq n. \tag{1.9}$$

Let $\mathcal{P}_n$ be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=n}^{\infty} c_k z^k \quad (c_n \neq 0) \tag{1.10}$$

which are analytic in $\mathbb{U}$. We also denote by $\mathcal{Q}_n$ the subclass of $\mathcal{P}_n$ consisting of $f(z)$ which satisfy

$$|p(z) - 1| < 1 \quad (z \in \mathbb{U}). \tag{1.11}$$

Since $p(z) \in \mathcal{Q}_n$ shows that $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$, $p(z) \in \mathcal{Q}_n$ is said to be a Carathéodory function in $\mathbb{U}$ (see Carathéodory [1]).

2. Conditions for the classes $\mathcal{Q}_n$ and $\mathcal{T}_n^*$. Applying Lemma 1.1, we discuss some conditions for $p(z) \in \mathcal{P}_n$ to be in the class $\mathcal{Q}_n$.

Theorem 2.1. *If $p(z) \in \mathcal{P}_n$ satisfies*

$$\operatorname{Re}\left(p(z) + \alpha \frac{zp'(z)}{p(z)}\right) < \sqrt{\alpha n}|p(z)| \quad (z \in \mathbb{U}) \tag{2.1}$$

for some real $\alpha > 0$, then $p(z) \in \mathcal{Q}_n$.

Proof. Note that $p(z) \neq 0$ $(z \in \mathbb{U})$ with the condition (2.1). Let us define the function $w(z)$ by

$$p(z) = 1 + w(z) \quad (z \in \mathbb{U}) \tag{2.2}$$

for $p(z) \in \mathcal{P}_n$. Then $w(z)$ is analytic in $\mathbb{U}$ and

$$w(z) = c_n z^n + c_{n+1} z^{n+1} + \ldots. \tag{2.3}$$

It follows that

$$p(z) + \alpha \frac{zp'(z)}{p(z)} = 1 + w(z) + \frac{\alpha z w'(z)}{1 + w(z)} \tag{2.4}$$

and that

$$\begin{aligned} &\frac{1}{|p(z)|} \operatorname{Re}\left(p(z) + \alpha \frac{zp'(z)}{p(z)}\right) \\ &\qquad = \frac{1}{|1 + w(z)|} \operatorname{Re}\left(1 + w(z) + \frac{\alpha z w'(z)}{1 + w(z)}\right) < \sqrt{\alpha n} \end{aligned} \tag{2.5}$$

for $z \in \mathbb{U}$.

We suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1. \tag{2.6}$$

Then, Lemma 1.1 gives us that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = me^{i\theta}$ $(m \geq n)$. For such a point z_0, we have that

$$
\begin{aligned}
\frac{1}{|p(z_0)|} \operatorname{Re}\left(p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)}\right) &= \frac{1}{|1+e^{i\theta}|} \operatorname{Re}\left(1 + e^{i\theta} + \frac{\alpha m e^{i\theta}}{1+e^{i\theta}}\right) \\
&= \frac{1}{\sqrt{2(1+\cos\theta)}}\left(1 + \cos\theta + \frac{\alpha m}{2}\right) \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{1+\cos\theta} + \frac{\alpha m}{2\sqrt{1+\cos\theta}}\right) \\
&\geq \sqrt{\alpha m} \geq \sqrt{\alpha n}.
\end{aligned}
\tag{2.7}
$$

This contradicts the condition (2.1). Therefore, there is no such point $z_0 \in \mathbb{U}$. This means that $p(z) \in \mathcal{Q}_n$. □

Corollary 2.1. *If $f(z) \in \mathcal{A}_n$ satisfies $f(z)f'(z) \neq 0$ for $0 < |z| < 1$ and*

$$
\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} < \sqrt{\alpha n}\left|\frac{zf'(z)}{f(z)}\right| \quad (z \in \mathbb{U})
\tag{2.8}
$$

for some real $\alpha > 0$, then $f(z) \in \mathcal{T}_n^*$.

Proof. Letting $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2.1, we have that

$$
p(z) + \alpha \frac{zp'(z)}{p(z)} = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right).
$$

The proof of the corollary follows from the above. □

Next we derive

Theorem 2.2. *If $p(z) \in \mathcal{P}_n$ satisfies $\operatorname{Re} p(z) \neq 0$ $(z \in \mathbb{U})$ and*

$$
\operatorname{Re}\left(p(z) + \alpha \frac{zp'(z)}{p(z)}\right) < \left(1 + \frac{\alpha n}{4}\right) \operatorname{Re} p(z) \quad (z \in \mathbb{U})
\tag{2.9}
$$

for some real $\alpha > 0$, then $p(z) \in \mathcal{Q}_n$.

Proof. Define the function $w(z)$ by (2.2) for $p(z) \in \mathcal{P}_n$. Then, $w(z)$ is analytic in $\mathbb{U}$,

$$
w(z) = c_n z^n + c_{n+1} z^{n+1} + \dots,
$$

and

$$
\frac{\operatorname{Re}\left(p(z) + \alpha \frac{zp'(z)}{p(z)}\right)}{\operatorname{Re} p(z)} = \frac{\operatorname{Re}\left(1 + w(z) + \frac{\alpha z w'(z)}{1+w(z)}\right)}{\operatorname{Re}(1+w(z))} < 1 + \frac{\alpha n}{4}
\tag{2.10}
$$

$(z \in \mathbb{U})$. If we suppose that there exists a point $z_0 \in \mathbb{U}$ on the circle $|z| = r < 1$ such that

$$
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,
$$

we can write that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = me^{i\theta}$. This shows that

$$\text{(2.11)} \quad \frac{\operatorname{Re}\left(p(z_0) + \alpha \dfrac{z_0 p'(z_0)}{p(z_0)}\right)}{\operatorname{Re} p(z_0)} = \frac{1 + \cos\theta + \dfrac{\alpha m}{2}}{1 + \cos\theta} \geq 1 + \frac{\alpha m}{4} \geq 1 + \frac{\alpha n}{4}.$$

Since (2.11) contradicts our condition (2.9), $|w(z)| < 1$ for all $z \in \mathbb{U}$. This means that $p(z) \in \mathcal{Q}_n$. □

If we take $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2.2, we have

Corollary 2.2. *If* $f(z) \in \mathcal{A}_n$ *satisfies* $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \neq 0$ $(z \in \mathbb{U})$ *and*

$$\text{(2.12)} \quad \operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} < \left(1 + \frac{\alpha n}{4}\right)\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)$$

$(z \in \mathbb{U})$ for some real $\alpha > 0$, then $f(z) \in \mathcal{T}_n^*$.

Corollary 2.3. *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$\text{(2.13)} \quad \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) + \frac{n-2}{n} \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{T}_n^*$.

Proof. If we write

$$\frac{zf'(z)}{f(z)} = 1 + w(z) \quad (f(z) \in \mathcal{A}_n),$$

we see that $w(z)$ is analytic in $\mathbb{U}$ and

$$w(z) = c_n z^n + c_{n+1} z^{n+1} + \dots.$$

For such a function $w(z)$, we see that

$$\text{(2.14)} \quad \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{zw'(z)}{1 + w(z)} - 1\right) < \frac{n-2}{2} \quad (z \in \mathbb{U}).$$

Supposing that there exists a point $z_0 \in \mathbb{U}$ on the circle $|z| = r < 1$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

we can write that $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = me^{i\theta}$. Therefore, we have

$$\text{(2.15)} \quad \operatorname{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 f'(z_0)}{f(z_0)}\right) = \operatorname{Re}\left(\frac{ke^{i\theta}}{1 + e^{i\theta}} - 1\right) = \frac{k}{2} - 1 \geq \frac{n-2}{2},$$

which contradicts the condition (2.13). This implies that $f(z) \in \mathcal{T}_n^*$. □

Example 2.1. Let us consider the function $p(z)$ given by

$$p(z) = 1 + a_n z^n \quad (z \in \mathbb{U}) \tag{2.16}$$

for some $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, where a_n satisfies

$$a_n^3 + 2a_n - 1 \leq 0 \quad (0 < a_n < 1).$$

Then $p(z) \in \mathcal{P}_n$ and $p(z) \neq 0$ $(z \in \mathbb{U})$. It is clear that $p(z)$ satisfies the condition (2.9) in Theorem 2.2 for $z = 0$.

Let us put $z = e^{i\theta}$ for $p(z)$. Then we see that

$$\operatorname{Re}\left(p(z) + \alpha \frac{zp'(z)}{p(z)}\right) = 1 + a_n \cos n\theta + \frac{\alpha n a_n (a_n + \cos n\theta)}{a_n^2 + 1 + 2a_n \cos n\theta} \tag{2.17}$$

and

$$\left(1 + \frac{\alpha n}{4}\right) \operatorname{Re} p(z) = \left(1 + \frac{\alpha n}{4}\right)(1 + a_n \cos n\theta). \tag{2.18}$$

This gives us that

$$\begin{aligned} &\left(1 + \frac{\alpha n}{4}\right) \operatorname{Re} p(z) - \operatorname{Re}\left(p(z) + \alpha \frac{zp'(z)}{p(z)}\right) \\ &= \frac{\alpha n (1 + 2a_n \cos n\theta + a_n^3 \cos n\theta + 2a_n^2 \cos^2 n\theta)}{4(a_n^2 + 1 + 2a_n \cos n\theta)} \\ &\geq \frac{\alpha n (1 - 2a_n - a_n^3)}{4(a_n^2 + 1 + 2a_n \cos n\theta)} \geq 0. \end{aligned} \tag{2.19}$$

Therefore, the function $p(z)$ satisfies the condition (2.9) for all $z \in \mathbb{U}$. Indeed, we see that

$$|p(z) - 1| = |a_n z^n| < a_n < 1 \quad (z \in \mathbb{U}).$$

Furthermore, if we define the function $f(z) \in \mathcal{A}_n$ by

$$\frac{zf'(z)}{f(z)} = 1 + a_n z^n \tag{2.20}$$

with some real a_n $(0 < a_n < 1)$ satisfying

$$a_n^3 + 2a_n - 1 \leq 0,$$

then we have that

$$f(z) = z e^{\frac{a_n}{n} z^n} \tag{2.21}$$

which satisfies the condition (2.12) in Corollary 2.2.

If we consider the function

$$g(x) = x^3 + 2x - 1 \quad (0 < x < 1),$$

we see that $g(0) = -1 < 0$ and $g\left(\frac{1}{2}\right) = \frac{1}{8} > 0$. Therefore, there exists some real x $(0 < x < 1)$ such that $g(x) \leq 0$. Indeed, we see that

$$0.4533 < x < 0.4534.$$

3. Properties for the classes $\mathcal{P}_n$ and $\mathcal{A}_n$. We discuss some properties for functions in the classes $\mathcal{P}_n$ and $\mathcal{A}_n$.

Theorem 3.1. *If $p(z) \in \mathcal{P}_n$ satisfies*

$$\int_{|z|=r} \left| \operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right)\right| d\theta < \pi \tag{3.1}$$

for $z = re^{i\theta}$ $(0 < r < 1)$, then $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$.

Proof. It follows from (3.1) that

$$\int_{|z|=r} \left| \operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right)\right| d\theta = \int_0^{2\pi} \left|\frac{d \arg p(z)}{d\theta}\right| d\theta = \int_{|z|=r} |d \arg p(z)| < \pi. \tag{3.2}$$

This implies that $\operatorname{Re} p(z) > 0$ for $|z| = r < 1$. Applying the maximum principle for harmonic functions, we obtain that $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$. □

From Theorem 3.1, we have

Corollary 3.1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\int_{|z|=r} \left| \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| d\theta < \pi \tag{3.3}$$

for $z = re^{i\theta}$ $(0 < r < 1)$, then $f(z) \in \mathcal{S}_n^*$.

Further, applying the same method as the proof by Umezawa [5] and Nunokawa [3], we derive the following result:

Theorem 3.2. *If $f(z) \in \mathcal{A}_1$ satisfies*

$$-\frac{\beta}{4\beta - 1} < \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \beta \quad (z \in \mathbb{U}) \tag{3.4}$$

for some real $\beta \geq \frac{1}{4}$, then $\operatorname{Re} f'(z) > 0$ $(z \in \mathbb{U})$.

Proof. We note that if $f'(z_0) = 0$ for some $z_0 \in \mathbb{U}$, then $f(z)$ does not satisfy the condition (3.4). This shows that $f'(z) \neq 0$ for all $z \in \mathbb{U}$. Applying the same method by Umezawa [5] and Nunokawa [3], we have that

$$\int_{|z|=r} \frac{zf''(z)}{f'(z)} d\theta = \int_{|z|=r} \frac{zf''(z)}{f'(z)} \frac{dz}{iz} = -i \int_{|z|=r} \frac{zf''(z)}{f'(z)} dz = 0. \tag{3.5}$$

We denote by $\mathcal{C}_1$ the part of the circle $|z| = r$ on which

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) \geq 0 \tag{3.6}$$

and

$$\int_{\mathcal{C}_1} d \arg z = x. \tag{3.7}$$

On the other hand, let us denote by $\mathcal{C}_2$ the part of the circle $|z| = r$ on which

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) < 0 \tag{3.8}$$

and

$$\int_{\mathcal{C}_2} d\arg z = 2\pi - x. \tag{3.9}$$

Putting

$$y_1 = \int_{\mathcal{C}_1} \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) d\theta = \int_{\mathcal{C}_1} \left(\frac{d\arg f'(z)}{d\theta}\right) d\theta \tag{3.10}$$

and

$$-y_2 = \int_{\mathcal{C}_2} \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) d\theta = \int_{\mathcal{C}_2} \left(\frac{d\arg f'(z)}{d\theta}\right) d\theta, \tag{3.11}$$

we have that $y_1 - y_2 = 0$.

In view of the condition (3.4), we obtain that

$$y_1 < \beta x \quad \text{and} \quad y_2 < \frac{\beta}{4\beta - 1}(2\pi - x).$$

If $y_1 \geq \frac{\pi}{2}$, then $y_2 = y_1 \geq \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta x$. On the other hand, we have that

$$y_2 < \frac{\beta}{4\beta - 1}(2\pi - x) < \frac{2\pi\beta - \frac{\pi}{2}}{4\beta - 1} = \frac{\pi}{2}. \tag{3.12}$$

This contradicts the inequality $y_2 \geq \frac{\pi}{2}$. Therefore, $y_1 = y_2 < \frac{\pi}{2}$. Consequently, we obtain that

$$y_1 + y_2 = \int_{|z|=r} \left|\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right| d\theta = \int_{|z|=r} |d\arg f'(z)| < \pi, \tag{3.13}$$

which implies that $\operatorname{Re} f'(z) > 0$ $(z \in \mathbb{U})$. □

Finally, letting $\beta \to \infty$, $\beta = \frac{1}{4}$ and $\beta = \frac{1}{2}$ in Theorem 3.2, we have the following corollary.

Corollary 3.2. *If $f(z) \in \mathcal{A}_1$ satisfies one of the following conditions*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > -\frac{1}{4} \quad (z \in \mathbb{U}), \tag{3.14}$$

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \frac{1}{4} \quad (z \in \mathbb{U}), \tag{3.15}$$

$$\left|\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right| < 1 \quad (z \in \mathbb{U}), \tag{3.16}$$

then $\operatorname{Re} f'(z) > 0$ $(z \in \mathbb{U})$.

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