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Strongly gamma-starlike functions of order alpha

Abstract. In this work we consider the class of analytic functions $\mathcal{G}(\alpha,\gamma)$, which is a subset of gamma-starlike functions introduced by Lewandowski, Miller and Złotkiewicz in *Gamma starlike functions*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **28** (1974), 53–58. We discuss the order of strongly starlikeness and the order of strongly convexity in this subclass.

1. Introduction. Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D}=\{z: \ |z|<1\}$ on the complex plane $\mathbb{C}$. For $a\in\mathbb{C}$ and $n\in\mathbb{N}$ we denote by

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}: f(z)=a+a_nz^n+\dots\}$$

and

$$\mathcal{A}_n=\left\{f\in\mathcal{H}: f(z)=z+a_{n+1}z^{n+1}+\dots\right\},$$

so $\mathcal{A}=\mathcal{A}_1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{D}$.

The class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha<1$ may be defined as

$$\mathcal{S}^*_\alpha=\left\{f\in\mathcal{A}: \ \mathfrak{Re}\frac{zf'(z)}{f(z)}>\alpha, \ z\in\mathbb{D}\right\}.$$

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The class $\mathcal{S}^*_\alpha$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha < 1$

$$\begin{aligned}\mathcal{K}_\alpha &:= \left\{ f \in \mathcal{A} : \ \mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{D} \right\} \\ &= \left\{ f \in \mathcal{A} : \ zf' \in \mathcal{S}^*_\alpha \right\}\end{aligned}$$

were introduced by Robertson in [6], see also [2]. If $\alpha \in [0;1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular we denote $\mathcal{S}^*_0 = \mathcal{S}^*$, $\mathcal{K}_0 = \mathcal{K}$, the classes of starlike and convex functions, respectively. Furthermore, note that if $f \in \mathcal{K}_\alpha$, then $f \in \mathcal{S}^*_{\delta(\alpha)}$, see [9], where

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}-2} & \text{for } \alpha \neq \frac{1}{2}, \\ \frac{1}{2\log 2} & \text{for } \alpha = \frac{1}{2}. \end{cases} \tag{1.1}$$

Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β, $0 < \beta < 2$,

$$\mathcal{SS}^*(\beta) := \left\{ f \in \mathcal{A} : \left| \mathrm{Arg} \frac{zf'(z)}{f(z)} \right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D} \right\}, \tag{1.2}$$

which was introduced in [8] and [1]. Furthermore,

$$\mathcal{SK}(\beta) = \left\{ f \in \mathcal{A} : \ zf' \in \mathcal{SS}^*(\beta) \right\}$$

denotes the class of strongly convex functions of order β. Analogously to (1.1), in the work [5] it was proved that if $\beta \in (0,1)$ and $f \in \mathcal{SK}(\alpha(\beta))$, then $f \in \mathcal{SS}(\beta)$, where

$$\alpha(\beta) = \beta + \frac{2}{\pi}\tan^{-1}\left(\frac{\beta n(\beta)\sin(\pi(1-\beta)/2)}{m(\beta) + \beta n(\beta)\cos(\pi(1-\beta)/2)}\right), \tag{1.3}$$

and where

$$m(\beta) = (1+\beta)^{(1+\beta)/2}, \qquad n(\beta) = (1-\beta)^{(\beta-1)/2}.$$

The class $\mathcal{G}(\alpha, \gamma)$, $\gamma > 0$, $0 < \alpha \leq 1$ of γ-strongly starlike functions of order α consists of functions $f \in \mathcal{A}$ satisfying

$$\left| \mathrm{Arg}\left\{ \left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma} \right\} \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}, \tag{1.4}$$

and such that

$$f(z)f'(z)\left(1 + \frac{zf''(z)}{f'(z)}\right) \neq 0, \quad z \in \mathbb{D} \setminus \{0\}. \tag{1.5}$$

Note that Lewandowski, Miller and Złotkiewicz, 1974 [3] have introduced the class of γ-starlike functions, denoted here by $\mathcal{G}(1, \gamma)$, which satisfy (1.5) and such that

$$\mathfrak{Re}\left\{ \left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma} \right\} > 0, \quad z \in \mathbb{D}. \tag{1.6}$$

2. Preliminaries. To prove the main results, we need the following Nunokawa's Lemma.

Lemma 2.1 ([4], [5]). *Let p be an analytic function in $|z| < 1$ with $p(0) = 1$, $p(z) \neq 0$. If there exists a point z_0, $|z_0| < 1$, such that*

$$|\operatorname{Arg}\{p(z)\}| < \frac{\pi\alpha}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$|\operatorname{Arg}\{p(z_0)\}| = \frac{\pi\alpha}{2}$$

for some $\alpha > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \operatorname{Arg}\{p(z_0)\} = \frac{\pi\alpha}{2}$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \operatorname{Arg}\{p(z_0)\} = -\frac{\pi\alpha}{2},$$

where

$$\{p(z_0)\}^{1/\alpha} = \pm ia, \quad \textit{and} \quad a > 0.$$

Moreover,

$$\operatorname{Arg}\left\{1 + \frac{z_0 p'(z_0)}{p^2(z_0)}\right\} \geq \tan^{-1}\left(\frac{\alpha n(\alpha)\sin(\pi(1-\alpha)/2)}{m(\alpha) + \alpha n(\alpha)\cos(\pi(1-\alpha)/2)}\right), \tag{2.1}$$

where

$$m(\alpha) = (1+\alpha)^{(1+\alpha)/2} \quad n(\alpha) = (1-\alpha)^{(\alpha-1)/2}.$$

3. Main result.

Theorem 3.1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be an analytic function in $\mathbb{D}$. Suppose also that $0 < \alpha \leq 1$ and γ is a positive real number such that f satisfies*

$$\left|\operatorname{Arg}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\}\right| < \frac{\alpha\pi}{2} \quad \textit{for} \quad |z| < 1. \tag{3.1}$$

If the equation, with respect to x,

$$x + \frac{2\gamma}{\pi}\tan^{-1}\left(\frac{xn(x)\sin(\pi(1-x)/2)}{m(x) + xn(x)\cos(\pi(1-x)/2)}\right) = \alpha, \tag{3.2}$$

where

$$m(x) = (1+x)^{(1+x)/2}, \qquad n(x) = (1-x)^{(x-1)/2},$$

has a solution $\beta \in (0, 1]$, then f is strongly starlike of order β.

Proof. Let us put

$$p(z)=\frac{zf'(z)}{f(z)},\ p(0)=1,\ (z\in\mathbb{D}). \tag{3.3}$$

Then we have

$$f(z)f'(z)\left(1+\frac{zf''(z)}{f'(z)}\right)\neq 0\quad\text{for}\quad 0<|z|<1$$

because of the assumption (3.1). Moreover,

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}=p(z)\left(1+\frac{zp'(z)}{p^2(z)}\right)^{\gamma}. \tag{3.4}$$

If there exists a point z_0, $|z_0|<1$, such that

$$|\mathrm{Arg}\,\{p(z)\}\,|<\frac{\pi\beta}{2}\quad\text{for}\quad |z|<|z_0|$$

and

$$|\mathrm{Arg}\,\{p(z_0)\}\,|=\frac{\pi\beta}{2},$$

then by Nunokawa's Lemma 2.1, we have

$$\frac{z_0p'(z_0)}{p(z_0)}=i\beta k,$$

where

$$k\geq 1\quad\text{when}\quad \mathrm{Arg}\,\{p(z_0)\}=\frac{\pi\beta}{2}$$

and

$$k\leq -1\quad\text{when}\quad \mathrm{Arg}\,\{p(z_0)\}=-\frac{\pi\beta}{2}.$$

For the case $\mathrm{Arg}\,\{p(z_0)\}=\pi\beta/2$, we have from (3.4) and (2.1)

$$\begin{aligned}\mathrm{Arg}&\left\{\left(\frac{z_0f'(z_0)}{f(z_0)}\right)^{1-\gamma}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)^{\gamma}\right\}\\&=\mathrm{Arg}\,\{p(z_0)\}+\gamma\mathrm{Arg}\left\{1+\frac{z_0p'(z_0)}{p^2(z_0)}\right\}\\&\geq\frac{\pi\beta}{2}+\gamma\tan^{-1}\left(\frac{\beta n(\beta)\sin(\pi(1-\beta)/2)}{m(\beta)+\beta n(\beta)\cos(\pi(1-\beta)/2)}\right)\\&=\frac{\alpha\pi}{2}\end{aligned}$$

because β is the solution of (3.2). For the case $\mathrm{Arg}\,\{p(z_0)\}=-\pi\beta/2$, applying the same method as the above, we have

$$\mathrm{Arg}\left\{\left(\frac{z_0f'(z_0)}{f(z_0)}\right)^{1-\gamma}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)^{\gamma}\right\}<-\frac{\alpha\pi}{2}.$$

In both of the above cases we have

$$\left|\operatorname{Arg}\left\{\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)^{1-\gamma}\left(1+\frac{z_0 f''()z)}{f'(z_0)}\right)^{\gamma}\right\}\right| \geq \frac{\alpha\pi}{2}$$

for $z_0 \in \mathbb{D}$, which contradicts hypothesis (3.1) of the theorem and therefore,

$$|\operatorname{Arg}\{p(z)\}| < \frac{\pi\beta}{2} \quad \text{for} \quad |z| < 1,$$

which completes the proof. □

Theorem 3.1 says that a function in the class $\mathcal{G}(\alpha, \gamma)$, see (1.4), of γ-strongly starlike functions of order α is strongly starlike function, see (1.2), of order at least β, where β is the solution of (3.2). Note that if $f \in \mathcal{G}(\alpha, 0)$, then f is strongly starlike of order α. For a related result we refer to [7]. If $\alpha = 1$, then Theorem 3.1 becomes the following result on the class $\mathcal{G}(\gamma, 1)$ introduced by Lewandowski, Miller and Złotkiewicz [3].

Corollary 3.1. *Assume that $f \in \mathcal{G}(\gamma, 1)$ or that f satisfies* (1.5) *and* (1.6). *If the equation*

$$x + \frac{2\gamma}{\pi}\tan^{-1}\left(\frac{xn(x)\sin(\pi(1-x)/2)}{m(x) + xn(x)\cos(\pi(1-x)/2)}\right) = 1,$$

has a solution $\beta \in (0, 1]$, then f is strongly starlike of order β.

In the corollary below there are examples of the choice α, γ and β which satisfies Corollary 3.1 or Theorem 3.1.

Corollary 3.2. *If $f \in \mathcal{G}(\gamma_{(1;1/2)}, 1)$, then $f \in \mathcal{SS}^*(1/2)$, where*

$$\gamma_{(1;1/2)} = \frac{\pi}{4\tan^{-1}\left(\frac{1}{3\sqrt[4]{4/3}+1}\right)} \approx 3.378.$$

If $f \in \mathcal{G}(\gamma_{(3/4;1/2)}, 3/4)$, then $f \in \mathcal{SS}^*(1/2)$, where

$$\gamma_{(3/4;1/2))} = \frac{\pi}{8\tan^{-1}\left(\frac{1}{3\sqrt[4]{4/3}+1}\right)} \approx 1.689.$$

If $f \in \mathcal{G}(\gamma_{(3/5;1/2)}, 3/5)$, then $f \in \mathcal{SS}^*(1/2)$, where

$$\gamma_{(3/5;1/2)} = \frac{\pi}{20\tan^{-1}\left(\frac{1}{3\sqrt[4]{4/3}+1}\right)} \approx 0.675.$$

If $f \in \mathcal{G}(\gamma_{(3/4;2/3)}, 3/4)$, then $f \in \mathcal{SS}^*(2/3)$, where

$$\gamma_{(3/4;2/3)} = \frac{\pi}{8\tan^{-1}\left(\frac{\sqrt[6]{3}}{3(5/3)^{5/3}+\sqrt[6]{81}}\right)} \approx 1.481.$$

If $\alpha = \beta$, then from (3.2) we get $\gamma = 0$, and $\mathcal{G}(0,\alpha) \subset \mathcal{SS}^*(\alpha)$, but this case is trivial. In the next theorem we consider the order of strongly starlikeness for functions, in some sense, in the class $\mathcal{G}(\alpha,\gamma)$ of negative order γ.

Theorem 3.2. *Let $f(z) = z + a_2z^2 + a_3z^3 + \dots$ be an analytic function in $\mathbb{D}$. Suppose also that γ is a negative real number such that f satisfies*

$$\left|\mathrm{Arg}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\}\right| < \frac{\alpha\pi}{2} \quad for \quad |z|<1, \tag{3.5}$$

where $0 < \alpha \le 1$ and suppose that β is the root of the equation

$$\beta + \gamma(1-\beta) = \alpha \tag{3.6}$$

in the interval $(0,1]$. Then we have

$$\left|\mathrm{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\beta\pi}{2} \quad for \quad |z|<1.$$

Proof. In the first part of the proof we apply the same method as in the proof of Theorem 3.1. Let us put

$$p(z) = \frac{zf'(z)}{f(z)},\ p(0)=1,\ (z\in\mathbb{D}).$$

If there exists a point $z_0 \in \mathbb{D}$ such that

$$\left|\mathrm{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi\beta}{2} \quad \text{for} \quad |z|<|z_0| \tag{3.7}$$

and

$$\left|\mathrm{Arg}\left\{\frac{z_0f'(z_0)}{f(z_0)}\right\}\right| = \frac{\pi\beta}{2},$$

then by Nunokawa's Lemma 2.1, we have

$$\frac{z_0p'(z_0)}{p(z_0)} = i\beta k, \tag{3.8}$$

where

$$k \ge \frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad \mathrm{Arg}\,\{p(z_0)\} = \frac{\pi\alpha}{2}$$

and

$$k \le -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad \mathrm{Arg}\,\{p(z_0)\} = -\frac{\pi\alpha}{2},$$

where

$$\{p(z_0)\}^{1/\beta} = \pm ia, \quad \text{and} \quad a>0. \tag{3.9}$$

For the case $\{p(z_0)\}^{1/\beta} = ia$, $a > 0$, we have from (3.8) and (3.9)

$$\begin{aligned}
&\operatorname{Arg}\left\{\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)^{1-\gamma}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right)^{\gamma}\right\} \\
&= \operatorname{Arg}\left\{p(z_0)\left(1+\frac{z_0 p'(z_0)}{p^2(z_0)}\right)^{\gamma}\right\} \\
&= \operatorname{Arg}\{p(z_0)\} + \gamma \operatorname{Arg}\left\{1+\frac{z_0 p'(z_0)}{p^2(z_0)}\right\} \\
&= \frac{\pi\beta}{2} + \gamma \operatorname{Arg}\left\{1+\frac{i\beta k}{(ia)^{\beta}}\right\} \\
&= \frac{\pi\beta}{2} + \gamma \operatorname{Arg}\left\{1+\frac{\beta k}{a^{\beta}} e^{i\frac{\pi(1-\beta)}{2}}\right\} \\
&\geq \frac{\pi\beta}{2} + \gamma \frac{\pi(1-\beta)}{2} \\
&= \frac{\alpha\pi}{2},
\end{aligned}$$

because β is the solution of (3.6). For the case $\operatorname{Arg}\{p(z_0)\} = -\pi\beta/2$, applying the same method as the above, we have

$$\operatorname{Arg}\left\{\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)^{1-\gamma}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right)^{\gamma}\right\} \leq -\frac{\alpha\pi}{2}.$$

The above cases show that

$$\left|\operatorname{Arg}\left\{\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)^{1-\gamma}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right)^{\gamma}\right\}\right| \geq \frac{\alpha\pi}{2}, \quad z_0 \in \mathbb{D},$$

which contradicts hypothesis (3.5) of the theorem and therefore,

$$|\operatorname{Arg}\{p(z)\}| < \frac{\pi\beta}{2} \quad \text{for} \quad |z| < 1$$

which completes the proof. □

Theorem 3.3. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be an analytic function in $\mathbb{D}$. Suppose also that $0 < \alpha \leq 1$ and $0 < \gamma \leq 1$ are such that f satisfies* (3.1). *If the equation* (3.2) *has a solution $\alpha_0 \in (0, 1]$, then f is strongly convex of order $\{(1-\gamma)\alpha_0 + \alpha\}/\gamma$.*

Proof.

$$\begin{aligned}
&\left|\operatorname{Arg}\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\}\right| - \left|\operatorname{Arg}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\right\}\right| \\
&\leq \left|\operatorname{Arg}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\}\right| < \frac{\alpha\pi}{2} \quad \text{for} \quad |z| < 1.
\end{aligned}$$

Then by Theorem 3.1 we have

$$\begin{aligned}
&\left|\operatorname{Arg}\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\}\right| \\
&\quad\le \left|\operatorname{Arg}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\right\}\right| + \frac{\alpha\pi}{2} \\
&\quad< \frac{\pi(1-\gamma)\alpha_0}{2} + \frac{\alpha\pi}{2},
\end{aligned}$$

and so

$$\left|\operatorname{Arg}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\pi\left\{(1-\gamma)\alpha_0+\alpha\right\}}{2\gamma} \quad \text{for} \quad |z|<1. \qquad \square$$

Theorem 3.4. *Assume that the equation* (3.2) *has a solution* α_0, $0<\alpha_0<\alpha\le 1$. *If* $0<\delta<\gamma$, *then* $\mathcal{G}(\alpha,\gamma)\subset\mathcal{G}(\alpha,\delta)$.

Proof. Let us suppose that f is a member of $\mathcal{G}(\alpha,\gamma)$ and let us put

$$A=\left\{B^{1-\delta}C^{\delta}\right\}^{\gamma/\delta},$$

where

$$B=zf'(z)/f(z) \quad \text{and} \quad C=1+zf''(z)/f'(z).$$

Then we have

$$A=B^{1-\gamma}C^{\gamma}B^{\gamma/\delta-1}$$

and by Theorem 3.1 we obtain

$$\begin{aligned}
|\operatorname{Arg}\{A\}| &= \frac{\gamma}{\delta}\left|\operatorname{Arg}\left\{B^{1-\delta}C^{\delta}\right\}\right| \\
&= \left|\operatorname{Arg}\left\{B^{1-\gamma}C^{\gamma}\right\}+\operatorname{Arg}\left\{B^{\gamma/\delta-1}\right\}\right| \\
&< \frac{\alpha\pi}{2}+\left(\frac{\gamma}{\delta}-1\right)\frac{\alpha_0\pi}{2} \\
&< \frac{\alpha\pi}{2}+\left(\frac{\gamma}{\delta}-1\right)\frac{\alpha\pi}{2} \\
&\le \frac{\gamma}{\delta}\frac{\alpha\pi}{2}.
\end{aligned}$$

This shows that

$$\left|\operatorname{Arg}\left\{B^{1-\delta}C^{\delta}\right\}\right| < \frac{\alpha\pi}{2} \quad z_0\in\mathbb{D}.$$

Therefore, $f\in\mathcal{G}(\alpha,\delta)$. $\square$

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