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Coefficient bounds for some subclasses of p-valently starlike functions

Abstract. For functions of the form $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ we obtain sharp bounds for some coefficients functionals in certain subclasses of starlike functions. Certain applications of our main results are also given. In particular, Fekete–Szegö-like inequality for classes of functions defined through extended fractional differintegrals are obtained.

1. Introduction. Let $\mathcal{A}_p$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic in the open disk $\triangle = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A} = \mathcal{A}_1$. For $f(z)$ given by (1.1) and $g(z)$ given by $g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}$, their *convolution* (or Hadamard product), denoted by $f * g$, is defined as

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}.$$

Given two functions f and g, which are analytic in $\triangle$, the function f is said to be *subordinate* to g in $\triangle$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\triangle$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in \triangle$. In particular, if the function g is univalent in $\triangle$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\triangle) \subset g(\triangle)$.

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Definition 1.1. Let $\phi(z)$ be an analytic function with positive real part in the unit disk $\triangle$ with $\phi(0) = 1$ and $\phi'(0) > 0$ that maps $\triangle$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class $\mathcal{M}_p(\lambda;\phi)$ is the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ satisfying

$$\frac{\frac{1}{p}\ zf'(z)}{(1-\lambda)z^p + \lambda f(z)} \prec \phi(z) \qquad (z \in \triangle,\ 0 < \lambda \leq 1). \tag{1.2}$$

As special cases, let

$$\mathcal{M}_p(1;\phi) = S_p^*(\phi) = \left\{ f(z) \in \mathcal{A}_p : \frac{1}{p}\frac{zf'(z)}{f(z)} \prec \phi(z) \right\},$$

$$\mathcal{M}_1(1;\phi) = S^*(\phi) = \left\{ f(z) \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}.$$

When $\phi(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, we denote the subclass $\mathcal{M}_p(\lambda;\phi)$ by $\mathcal{M}_p(\lambda;A,B)$. The class $\mathcal{M}_1(1;A,B) = S^*[A,B]$ was studied by Janowski [2]. For $0 \leq \alpha < 1$, let $\mathcal{M}_p(\lambda;\alpha) = \mathcal{M}_p(\lambda;1-2\alpha,-1)$.

For a fixed analytic function $g \in \mathcal{A}_p$ with positive coefficients, define the class $\mathcal{M}_{p,g}(\lambda;\phi)$ as the class of all functions $f \in \mathcal{A}_p$ satisfying $f * g \in \mathcal{M}_p(\lambda;\phi)$. This class includes as special cases several other classes studied in the literature. For example, when $g(z) = z^p + \sum_{n=1}^{\infty} \frac{p+n}{p} z^{p+n}$, the class $\mathcal{M}_{p,g}(1;\phi)$ reduces to the class $C_p(1:\phi) = C_p(\phi)$ consisting of functions $f \in \mathcal{A}_p$ satisfying

$$\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z), \qquad z \in \triangle. \tag{1.3}$$

The classes $S^*(\phi)$ and $C(\phi) = C_1(\phi)$ were introduced and studied by Ma and Minda [4]. They have obtained the Fekete–Szegö inequality for functions in the class $C(\phi)$. Since $f(z) \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete–Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of Fekete–Szegö problem for classes of starlike, convex and close-to-convex functions see the recent paper by Srivastava et al. [10].

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \ldots \tag{1.4}$$

in the unit disk $\triangle$ satisfying the condition $|w(z)| < 1$.

There has been triggering interest in the literature (see [1, 3, 4, 9, 10]) to define certain subclasses of analytic functions and to discuss Fekete–Szegö inequalities. Making use of the techniques, in this paper we defined two new classes $\mathcal{M}_p(\lambda;\phi)$ and $\mathcal{M}_{p,g}(\lambda;\phi)$ to obtain Fekete–Szegö inequalities and to discuss the results on upper bounds for the coefficient a_{p+3}.

Lemma 1.2 ([1]). *If $w \in \Omega$, then*

$$|w_2 - tw_1^2| \leq \begin{cases} -t & \text{if} \quad t \leq -1, \\ 1 & \text{if} \quad -1 \leq t \leq 1, \\ t & \text{if} \quad t \geq 1. \end{cases}$$

When $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = \frac{z(\lambda+z)}{1+\lambda z}$, $(0 \leq \lambda \leq 1)$ or one of its rotations while for $t = 1$, equality holds if and only if $w(z) = -\frac{z(\lambda+z)}{1+\lambda z}$, $(0 \leq \lambda \leq 1)$ or one of its rotations.

Also sharp upper bound above can be improved as follows when $-1 < t < 1$:

$$|w_2 - tw_1^2| + (1+t)|w_1|^2 \leq 1 \qquad (-1 < t \leq 0)$$

and

$$|w_2 - tw_1^2| + (1-t)|w_1|^2 \leq 1 \qquad (0 < t < 1).$$

Lemma 1.2 is a reformulation of Lemma of Ma and Minda [4].

Lemma 1.3 ([3]). *If $w \in \Omega$, then for any complex number t*

$$|w_2 - tw_1^2| \leq \max\{1, |t|\}.$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

Lemma 1.4 ([8]). *If $w \in \Omega$, then for any real numbers q_1 and q_2 the following sharp estimate holds:*

$$|w_3 + q_1 w_1 w_2 + q_2 w_1^3| \leq H(q_1, q_2)$$

where

$$H(q_1,q_2) = \begin{cases} 1 & \text{for } (q_1,q_2) \in D_1 \cup D_2 \\ |q_2| & \text{for } (q_1,q_2) \in \bigcup_{k=3}^{7} D_k \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_8 \cup D_9 \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{10} \cup D_{11} \backslash \{\pm 2, 1\} \\ \frac{2}{3}(|q_1-1|)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{12}. \end{cases}$$

The extremal functions, up to rotations, are of the form

$$w(z) = z^3, \quad w(z) = z, \quad w(z) = w_0(z) = \frac{(z[(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1] - \varepsilon_1\varepsilon_2 z)}{1 - [(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2]z},$$

$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z}, \quad w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z},$$

$$|\varepsilon_1| = |\varepsilon_2| = 1, \quad \varepsilon_1 = t_0 - e^{-\frac{i\theta_0}{2}}(a \mp b), \quad \varepsilon_2 = -e^{-\frac{i\theta_0}{2}}(ia \pm b),$$

$$a = t_0 \cos\frac{\theta_0}{2}, \quad b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}}, \quad \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left[\frac{2q_2(q_1^2+2) - 3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right]^{\frac{1}{2}}, \quad t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}}, \quad \cos\frac{\theta_0}{2} = \frac{q_1}{2}\left[\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right].$$

The sets $D_k,\ k = 1, 2, \ldots, 12$ *are defined as follows:*

$$D_1 = \left\{(q_1, q_2) : |q_1| \le \frac{1}{2}, |q_2| \le 1\right\},$$

$$D_2 = \left\{(q_1, q_2) : \frac{1}{2} \le |q_1| \le 2, \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \le q_2 \le 1\right\},$$

$$D_3 = \left\{(q_1, q_2) : |q_1| \le \frac{1}{2}, q_2 \le -1\right\},$$

$$D_4 = \left\{(q_1, q_2) : |q_1| \ge \frac{1}{2}, q_2 \le -\frac{2}{3}(|q_1|+1)\right\},$$

$$D_5 = \{(q_1, q_2) : |q_1| \le 2, q_2 \ge 1\},$$

$$D_6 = \left\{(q_1, q_2) : 2 \le |q_1| \le 4, q_2 \ge \frac{1}{12}(q_1^2+8)\right\},$$

$$D_7 = \left\{(q_1, q_2) : |q_1| \ge 4, q_2 \ge \frac{2}{3}(|q_1|-1)\right\},$$

$$D_8 = \left\{(q_1, q_2) : \frac{1}{2} \le |q_1| \le 2, -\frac{2}{3}(|q_1|+1) \le q_2 \le \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1)\right\},$$

$$D_9 = \left\{(q_1, q_2) : |q_1| \ge 2, -\frac{2}{3}(|q_1|+1) \le q_2 \le \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$

$$D_{10} = \left\{(q_1, q_2) : 2 \le |q_1| \le 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{1}{12}(q_1^2+8)\right\},$$

$$D_{11} = \left\{(q_1, q_2) : |q_1| \ge 4, \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\},$$

$$D_{12} = \left\{(q_1, q_2) : |q_1| \ge 4, \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \le q_2 \le \frac{2}{3}(|q_1|-1)\right\}.$$

2. Coefficient bounds. By making use of Lemmas 1.2–1.4, we prove the following bounds for the class $\mathcal{M}_p(\lambda;\phi)$.

Theorem 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$, *where* B_n*'s are real with* $B_1 > 0$ *and* $B_2 \geq 0$, *let* $0 < \lambda \leq 1$, *and*

$$\sigma_1 := \frac{[pB_1^2\lambda + (B_2 - B_1)(p - p\lambda + 1)](p - p\lambda + 1)}{(p - p\lambda + 2)pB_1^2},$$

$$\sigma_2 := \frac{[pB_1^2\lambda + (B_2 + B_1)(p - p\lambda + 1)](p - p\lambda + 1)}{(p - p\lambda + 2)pB_1^2},$$

$$\sigma_3 := \frac{[pB_1^2\lambda + B_2(p - p\lambda + 1)](p - p\lambda + 1)}{(p - p\lambda + 2)pB_1^2}.$$

If $f(z)$ *given by* (1.1) *belongs to* $\mathcal{M}_p(\lambda;\phi)$, *then*

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} -\dfrac{pB_1\Lambda}{p - p\lambda + 2} & \text{if} \quad \mu \leq \sigma_1 \\ \dfrac{pB_1}{p - p\lambda + 2} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2 \\ \dfrac{pB_1\Lambda}{p - p\lambda + 2} & \text{if} \quad \mu \geq \sigma_2. \end{cases} \tag{2.1}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{(p - p\lambda + 1)^2}{pB_1(p - p\lambda + 2)}(1 + \Lambda)|a_{p+1}|^2 \leq \frac{pB_1}{p - p\lambda + 2}. \tag{2.2}$$

If $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{(p - p\lambda + 1)^2}{pB_1(p - p\lambda + 2)}(1 - \Lambda)|a_{p+1}|^2 \leq \frac{pB_1}{p - p\lambda + 2}, \tag{2.3}$$

where

$$\Lambda = \frac{\mu(p - p\lambda + 2)pB_1^2 - \lambda(p - p\lambda + 1)pB_1^2 - (p - p\lambda + 1)^2 B_2}{(p - p\lambda + 1)^2 B_1}.$$

For any complex number μ,

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{pB_1}{p - p\lambda + 2}\max\{1, |\Lambda|\} \tag{2.4}$$

Further,

$$|a_{p+3}| \leq \frac{pB_1}{p - p\lambda + 3} H(q_1, q_2), \tag{2.5}$$

where $H(q_1, q_2)$ *is as defined in Lemma 1.3,*

$$q_1 := \frac{2B_2}{B_1} - \frac{(2p\lambda - 2p - 3)\lambda p B_1}{(p - p\lambda + 1)(p - p\lambda + 2)}$$

and

$$q_2 := \frac{B_3}{B_1} - \frac{(2p\lambda - 2p - 3)\lambda p B_2 - \lambda^2 p^2 B_1^2}{(p - p\lambda + 1)(p - p\lambda + 2)}.$$

These results are sharp.

Proof. If $f(z) \in \mathcal{M}_p(\lambda; \phi)$, then there is an analytic function $w(z)$ given by (1.4) such that

$$\frac{\frac{1}{p} z f'(z)}{(1-\lambda)z^p + \lambda f(z)} = \phi(w(z)). \tag{2.6}$$

Since

$$\begin{aligned}\frac{\frac{1}{p} z f'(z)}{(1-\lambda)z^p + \lambda f(z)} &= 1 + \left[\frac{1}{p}(p+1) - \lambda\right] a_{p+1} z + \left[\left(\frac{1}{p}(p+2) - \lambda\right) a_{p+2}\right. \\ &\quad \left. + \left(\lambda^2 - \frac{\lambda}{p}(p+1)\right) a_{p+1}^2\right] z^2 + \left[\left(\frac{1}{p}(p+3) - \lambda\right) a_{p+3}\right. \\ &\quad + \left(2\lambda^2 - \frac{\lambda}{p}(p+2) - \frac{\lambda}{p}(p+1)\right) a_{p+1} a_{p+2} \\ &\quad \left. + \left(\frac{1}{p}(p+1)\lambda^2 - \lambda^3\right) a_{p+1}^3\right] z^3 + \ldots\end{aligned}$$

and

$$\phi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + (B_1 w_3 + 2B_2 w_1 w_2 + B_3 w_1^3) z^3 + \ldots,$$

we have from (2.6),

$$a_{p+1} = \frac{p B_1 w_1}{p - p\lambda + 1}, \tag{2.7}$$

$$a_{p+2} = \frac{p(B_1 w_2 + B_2 w_1^2)}{p - p\lambda + 2} + \frac{\lambda p^2 B_1^2 w_1^2}{(p - p\lambda + 1)(p - p\lambda + 2)} \tag{2.8}$$

and

$$\begin{aligned} a_{p+3} = \frac{p B_1}{p - p\lambda + 3} \Bigg\{ w_3 &+ \left[\frac{2B_2}{B_1} - \frac{(2p\lambda - 2p - 3)\lambda p B_1}{(p - p\lambda + 1)(p - p\lambda + 2)}\right] w_1 w_2 \\ &+ \left[\frac{B_3}{B_1} - \frac{(2p\lambda - 2p - 3)\lambda p B_2 - \lambda^2 p^2 B_1^2}{(p - p\lambda + 1)(p - p\lambda + 2)}\right] w_1^3 \Bigg\}. \end{aligned} \tag{2.9}$$

Now,

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p B_1}{p - p\lambda + 2}\{w_2 - \Lambda w_1^2\}. \tag{2.10}$$

The results (2.1)–(2.3) are established by an application of Lemma 1.2, inequality (2.4) by Lemma 1.3 and (2.5) follows from Lemma 1.4. To show that the bounds in (2.1), (2.2) and (2.3) are sharp, we define $K_{\phi n}$, $n = 2, 3, 4, \dots$ by

$$\frac{\frac{1}{p} z K'_{\phi n}(z)}{(1-\lambda)z^p + \lambda K_{\phi n}(z)} = \phi(z^{n-1}), \qquad K_{\phi n}(0) = 0 = K'_{\phi n}(0) - 1$$

and the functions F_λ and G_λ, $0 < \lambda \leq 1$ by

$$\frac{\frac{1}{p} z F'_{\lambda}(z)}{(1-\lambda)z^p + \lambda F_{\lambda}(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \qquad F_\lambda(0) = 0 = F'_\lambda(0) - 1$$

and

$$\frac{\frac{1}{p} z G'_{\lambda}(z)}{(1-\lambda)z^p + \lambda G_{\lambda}(z)} = \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right), \qquad G_\lambda(0) = 0 = G'_\lambda(0) - 1.$$

Clearly the functions $K_{\phi n},\ F_\lambda,\ G_\lambda \in \mathcal{M}_p(\lambda; \phi)$.

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if f is $K_{\phi 2}$ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$ then equality holds if and only if f is F_λ or one of its rotations. Equality holds for $\mu = \sigma_2$ if and only if f is G_λ or one of its rotations. □

Remark 2.2. For $\lambda = 1$, results (2.1)–(2.4) coincide with the results obtained for the class $S_p^*(\phi)$ by Ali et al. [1].

Remark 2.3. For $\lambda = 1,\ p = 1$, results (2.1)–(2.4) coincide with the results obtained for the class $S^*(\phi)$ by Ma and Minda [4].

Example 2.4. Let $-1 \leq B < A \leq 1$. If $f(z)$ given by (1.1) belongs to $\mathcal{M}_p(\lambda; A, B)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} -\dfrac{p(A-B)\Lambda}{p-p\lambda+2} & \text{if} \quad \mu \leq \sigma_1 \\ \dfrac{p(A-B)}{p-p\lambda+2} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2 \\ \dfrac{p(A-B)\Lambda}{p-p\lambda+2} & \text{if} \quad \mu \geq \sigma_2. \end{cases}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{(p-p\lambda+1)^2}{p(A-B)(p-p\lambda+2)}(1+\Lambda)|a_{p+1}|^2 \leq \frac{p(A-B)}{p-p\lambda+2}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{(p-p\lambda+1)^2}{p(A-B)(p-p\lambda+2)}(1-\Lambda)|a_{p+1}|^2 \leq \frac{p(A-B)}{p-p\lambda+2},$$

where

$$\sigma_1 := \frac{[p(A-B)\lambda - (1+B)(p-p\lambda+1)](p-p\lambda+1)}{p(p-p\lambda+2)(A-B)},$$
$$\sigma_2 := \frac{[p(A-B)\lambda + (1-B)(p-p\lambda+1)](p-p\lambda+1)}{p(p-p\lambda+2)(A-B)},$$
$$\sigma_3 := \frac{[p(A-B)\lambda - B(p-p\lambda+1)](p-p\lambda+1)}{p(p-p\lambda+2)(A-B)}$$

and

$$\Lambda_A = \frac{\mu p(p-p\lambda+2)(A-B) - (p-p\lambda+1)[(A-B)\lambda p - (p-p\lambda+1)B]}{(p-p\lambda+1)^2}.$$

For any complex number μ,

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{p(A-B)}{p-p\lambda+2}\max\{1, |\Lambda_A|\}.$$

In particular, if $f \in \mathcal{M}_p(\lambda;\alpha)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} -\dfrac{2p(1-\alpha)\Lambda}{p-p\lambda+2} & \text{if} \quad \mu \leq \sigma_1 \\ \dfrac{2p(1-\alpha)}{p-p\lambda+2} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2 \\ \dfrac{2p(1-\alpha)\Lambda}{p-p\lambda+2} & \text{if} \quad \mu \geq \sigma_2. \end{cases}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{(p-p\lambda+1)^2}{2p(1-\alpha)(p-p\lambda+2)}(1+\Lambda)|a_{p+1}|^2 \leq \frac{2p(1-\alpha)}{p-p\lambda+2}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{(p-p\lambda+1)^2}{2p(1-\alpha)(p-p\lambda+2)}(1-\Lambda)|a_{p+1}|^2 \leq \frac{2p(1-\alpha)}{p-p\lambda+2},$$

where

$$\sigma_1 := \frac{\lambda(p-p\lambda+1)}{p-p\lambda+2},$$
$$\sigma_2 := \frac{(p-p\lambda\alpha+1)(p-p\lambda+1)}{p(1-\alpha)(p-p\lambda+2)},$$
$$\sigma_3 := \frac{[2p(1-\alpha)\lambda + (p-p\lambda+1)](p-p\lambda+1)}{2p(1-\alpha)(p-p\lambda+2)}$$

and

$$\Lambda_\alpha = \frac{2\mu p(1-\alpha)(p-p\lambda+2) - (p-p\lambda+1)[2\lambda p(1-\alpha) + (p-p\lambda+1)]}{(p-p\lambda+1)^2}.$$

For any complex number μ,

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{2p(1-\alpha)}{p - p\lambda + 2} \max\{1, |\Lambda_\alpha|\}.$$

The results are sharp.

Corollary 2.5. *Let $\phi(z)$ be as in Theorem 2.1,*

$$g(z) = z^p + \sum_{n=1}^{\infty} g_{p+n} z^{p+n} \quad (g_{p+n} > 0),$$

and let

$$\sigma_1 := \frac{g_{p+1}^2}{g_{p+2}} \frac{[pB_1^2\lambda + (B_2 - B_1)(p - p\lambda + 1)](p - p\lambda + 1)}{(p - p\lambda + 2)pB_1^2},$$
$$\sigma_2 := \frac{g_{p+1}^2}{g_{p+2}} \frac{[pB_1^2\lambda + (B_2 + B_1)(p - p\lambda + 1)](p - p\lambda + 1)}{(p - p\lambda + 2)pB_1^2},$$
$$\sigma_3 := \frac{g_{p+1}^2}{g_{p+2}} \frac{[pB_1^2\lambda + B_2(p - p\lambda + 1)](p - p\lambda + 1)}{(p - p\lambda + 2)pB_1^2}.$$

If $f(z)$ given by (1.1) *belongs to $\mathcal{M}_{p,g}(\lambda; \phi)$, then*

$$(2.11) \qquad |a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} -\dfrac{pB_1\Lambda}{g_{p+2}(p - p\lambda + 2)} & \text{if} \quad \mu \leq \sigma_1 \\ \dfrac{pB_1}{g_{p+2}(p - p\lambda + 2)} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2 \\ \dfrac{pB_1\Lambda}{g_{p+2}(p - p\lambda + 2)} & \text{if} \quad \mu \geq \sigma_2. \end{cases}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$(2.12) \qquad |a_{p+2} - \mu a_{p+1}^2| + \frac{g_{p+1}^2(p - p\lambda + 1)^2}{g_{p+2}(p - p\lambda + 2)pB_1}(1 + \Lambda)|a_{p+1}|^2 \leq \frac{pB_1}{g_{p+2}(p - p\lambda + 2)}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$(2.13) \qquad |a_{p+2} - \mu a_{p+1}^2| + \frac{g_{p+1}^2(p - p\lambda + 1)^2}{g_{p+2}(p - p\lambda + 2)pB_1}(1 - \Lambda)|a_{p+1}|^2 \leq \frac{pB_1}{g_{p+2}(p - p\lambda + 2)},$$

where

$$\Lambda_g = \frac{\frac{g_{p+2}}{g_{p+1}^2}\mu(p - p\lambda + 2)pB_1^2 - \lambda(p - p\lambda + 1)pB_1^2 - (p - p\lambda + 1)^2 B_2}{(p - p\lambda + 1)^2 B_1}.$$

For any complex number μ,

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{pB_1}{g_{p+2}(p - p\lambda + 2)} \max\{1, |\Lambda_g|\}.$$

Further,

$$|a_{p+3}| \leq \frac{pB_1}{g_{p+3}(p - p\lambda + 3)} H(q_1, q_2),$$

where $H(q_1, q_2)$ *is as defined in Lemma 1.3,*

$$q_1 := \frac{2B_2}{B_1} - \frac{(2p\lambda - 2p - 3)\lambda p B_1}{(p - p\lambda + 1)(p - p\lambda + 2)}$$

and

$$q_2 := \frac{B_3}{B_1} - \frac{(2p\lambda - 2p - 3)\lambda p B_2 - \lambda^2 p^2 B_1^2}{(p - p\lambda + 1)(p - p\lambda + 2)}.$$

These results are sharp.

3. Applications to functions defined by extended fractional differintegrals. With a view to define fractional differintegral operator $\Omega_z^{(\delta,p)}$, we recall Gauss hypergeometric function ${}_2F_1$ defined by [6].

$$(3.1) \quad {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad (a, b, c \in \mathbb{C},\ c \neq 0, -1, -2, \ldots),$$

where $(d)_n$ denotes the Pochhammer symbol given in terms of Gamma function Γ by

$$(d)_n = \frac{\Gamma(d+n)}{\Gamma(d)} = \begin{cases} 1 & (n = 0;\ d \in \mathbb{C} \setminus \{0\}) \\ d(d+1)\ldots(d+n-1) & (n \in \mathbb{N};\ d \in \mathbb{C}). \end{cases}$$

We note that the series defined by (3.1) converges absolutely for $z \in \triangle$ and hence ${}_2F_1$ represents an analytic function in $\triangle$.

Also we recall the definitions of fractional calculus considered by Owa [5] (see also [6, 11, 12]).

Definition 3.1. The fractional integral of order δ ($\delta > 0$) is defined, for a function f, analytic in a simply connected region of the complex plane containing the origin, by

$$(3.2) \qquad D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\delta}} d\zeta,$$

where the multiplicity of $(z - \zeta)^{\delta - 1}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

Definition 3.2. Under the hypothesis of Definition 3.1, the fractional derivative of f of order δ $(\delta \geq 0)$ is defined by

$$(3.3)\qquad D_z^\delta f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\delta)} \dfrac{d}{dz} \displaystyle\int_0^z \dfrac{f(\zeta)}{(z-\zeta)^\delta} d\zeta, & (0 \leq \delta < 1) \\ \dfrac{d^n}{dz^n} D_z^{\delta-n} f(z), & (n \leq \delta < n+1;\ n \in \mathbb{N}_0) \end{cases}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the multiplicity of $(z-\zeta)^{-\delta}$ is removed as in Definition 3.1.

Definition 3.3 ([7]). The extended fractional differintegral operator $\Omega_z^{(\delta,p)} : \mathcal{A}_p \to \mathcal{A}_p$ for a function f of the form (1.1) and for a real number δ $(-\infty < \delta < p+1)$ is defined by

$$(3.4)\qquad \Omega_z^{(\delta,p)} f(z) = \frac{\Gamma(p+1-\delta)}{\Gamma(p+1)} z^\delta D_z^\delta f(z) \quad (-\infty < \delta < p+1;\ z \in \triangle),$$

where $D_z^\delta f(z)$ is respectively, the fractional integral of f of order $-\delta$ when $-\infty < \delta < 0$ and the fractional derivative of f of order δ when $0 \leq \delta < p+1$.

We note that

$$\begin{aligned} \Omega_z^{(\delta,p)} f(z) &= z^p + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+1-\delta)}{\Gamma(p+1)\Gamma(n+p+1-\delta)} a_{n+p} z^{n+p} \\ &= z^p\ {}_2F_1(1, p+1; p+1-\delta; z) * f(z) \end{aligned}$$

$(-\infty < \delta < p+1;\ z \in \triangle)$.

Let $\mathcal{M}_{p,\delta}(\lambda;\phi)$ be the class of functions $f \in \mathcal{A}_p$ for which $\Omega_z^{(\delta,p)} f(z) \in \mathcal{M}_p(\lambda;\phi)$. The class $\mathcal{M}_{p,\delta}(\lambda;\phi)$ is the special case of the class $\mathcal{M}_{p,g}(\lambda;\phi)$, when

$$g(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+1-\delta)}{\Gamma(p+1)\Gamma(n+p+1-\delta)} z^{n+p}.$$

Since

$$\left(\Omega_z^{(\delta,p)} f\right)(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p+1-\delta)}{\Gamma(p+1)\Gamma(n+p+1-\delta)} a_{n+p} z^{n+p},$$

we have

$$(3.5)\qquad g_{p+1} = \frac{\Gamma(p+2)\Gamma(p+1-\delta)}{\Gamma(p+1)\Gamma(p+2-\delta)} = \frac{p+1}{p+1-\delta},$$

$$(3.6)\qquad g_{p+2} = \frac{\Gamma(p+3)\Gamma(p+1-\delta)}{\Gamma(p+1)\Gamma(p+3-\delta)} = \frac{(p+1)(p+2)}{(p+1-\delta)(p+2-\delta)},$$

$$(3.7)\qquad g_{p+3} = \frac{\Gamma(p+4)\Gamma(p+1-\delta)}{\Gamma(p+1)\Gamma(p+4-\delta)} = \frac{(p+1)(p+2)(p+3)}{(p+1-\delta)(p+2-\delta)(p+3-\delta)}.$$

For g_{p+1}, g_{p+2} and g_{p+3} given by (3.5), (3.6) and (3.7), Corollary 2.5 reduces to the following:

Theorem 3.4. *Let $\phi(z)$ be as in Theorem 2.1, and let*

$$\sigma_1 := \frac{(p+1)(p+2-\delta)}{(p+2)(p+1-\delta)}\frac{[pB_1^2\lambda + (B_2 - B_1)(p - p\lambda + 1)](p - p\lambda + 1)}{(p - p\lambda + 2)pB_1^2},$$
$$\sigma_2 := \frac{(p+1)(p+2-\delta)}{(p+2)(p+1-\delta)}\frac{[pB_1^2\lambda + (B_2 + B_1)(p - p\lambda + 1)](p - p\lambda + 1)}{(p - p\lambda + 2)pB_1^2},$$
$$\sigma_3 := \frac{(p+1)(p+2-\delta)}{(p+2)(p+1-\delta)}\frac{[pB_1^2\lambda + B_2(p - p\lambda + 1)](p - p\lambda + 1)}{(p - p\lambda + 2)pB_1^2}.$$

If $f(z)$ given by (1.1) *belongs to $\mathcal{M}_{p,\delta}(\lambda;\phi)$, then*

$$|a_{p+2} - \mu a_{p+1}^2| \le \begin{cases} -\dfrac{(p+1-\delta)(p+2-\delta)}{(p+1)(p+2)}\dfrac{pB_1\Lambda}{(p-p\lambda+2)} & \text{if } \mu \le \sigma_1, \\ \dfrac{(p+1-\delta)(p+2-\delta)}{(p+1)(p+2)}\dfrac{pB_1}{(p-p\lambda+2)} & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ \dfrac{(p+1-\delta)(p+2-\delta)}{(p+1)(p+2)}\dfrac{pB_1\Lambda}{(p-p\lambda+2)} & \text{if } \mu \ge \sigma_2. \end{cases}$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, then

$$\begin{aligned} &|a_{p+2} - \mu a_{p+1}^2| + \frac{(p+1)(p+2-\delta)}{(p+2)(p+1-\delta)}\frac{(p-p\lambda+1)^2}{(p-p\lambda+2)}(1+\Lambda)|a_{p+1}|^2 \\ &\qquad \le \frac{(p+1-\delta)(p+2-\delta)}{(p+1)(p+2)}\frac{pB_1}{(p-p\lambda+2)}. \end{aligned} \tag{3.8}$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\begin{aligned} &|a_{p+2} - \mu a_{p+1}^2| + \frac{(p+1)(p+2-\delta)}{(p+2)(p+1-\delta)}\frac{(p-p\lambda+1)^2}{(p-p\lambda+2)}(1+\Lambda)|a_{p+1}|^2 \\ &\qquad \le \frac{(p+1-\delta)(p+2-\delta)}{(p+1)(p+2)}\frac{pB_1}{(p-p\lambda+2)}, \end{aligned} \tag{3.9}$$

where

$$\Lambda_\delta = \frac{\frac{(p+2)(p+1-\delta)}{(p+1)(p+2-\delta)}\mu(p-p\lambda+2)pB_1^2 - \lambda(p-p\lambda+1)pB_1^2 - (p-p\lambda+1)^2B_2}{(p-p\lambda+1)^2B_1}.$$

For any complex number μ,

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{(p+1-\delta)(p+2-\delta)}{(p+1)(p+2)}\frac{pB_1}{(p-p\lambda+2)}\max\{1, |\Lambda_\delta|\}.$$

Further,

$$|a_{p+3}| \le \frac{(p+1-\delta)(p+2-\delta)(p+3-\delta)}{(p+1)(p+2)(p+3)}\frac{pB_1}{(p-p\lambda+3)}H(q_1,q_2),$$

where

$$q_1 := \frac{2B_2}{B_1} - \frac{(2p\lambda - 2p - 3)\lambda p B_1}{(p - p\lambda + 1)(p - p\lambda + 2)}$$

and

$$q_2 := \frac{B_3}{B_1} - \frac{(2p\lambda - 2p - 3)\lambda p B_2 - \lambda^2 p^2 B_1^2}{(p - p\lambda + 1)(p - p\lambda + 2)}.$$

These results are sharp.

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