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On certain subclasses of analytic functions associated with the Carlson–Shaffer operator

Abstract. The object of the present paper is to solve Fekete–Szegö problem and determine the sharp upper bound to the second Hankel determinant for a certain class $\mathscr{R}^\lambda(a, c, A, B)$ of analytic functions in the unit disk. We also investigate several majorization properties for functions belonging to a subclass $\widetilde{\mathscr{R}}^\lambda(a, c, A, B)$ of $\mathscr{R}^\lambda(a, c, A, B)$ and related function classes. Relevant connections of the main results obtained here with those given by earlier workers on the subject are pointed out.

1. Introduction and preliminaries. Let $\mathscr{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathscr{T}$ denote the subclass of $\mathscr{A}$ consisting of functions of the form

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geq 0). \tag{1.2}$$

A function $f \in \mathscr{A}$ is said to be starlike function of order α and convex function of order α, respectively, if and only if $\mathrm{Re}\{zf'(z)/f(z)\} > \alpha$ and $\mathrm{Re}\{1 + (zf''(z)/f'(z))\} > \alpha$ for $0 \leq \alpha < 1$ and for all $z \in \mathbb{U}$. By usual

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notations, we denote these classes of functions by $\mathscr{S}^\star(\alpha)$ and $\mathscr{C}(\alpha)$ $(0 \le \alpha < 1)$, respectively. We write $\mathscr{S}^\star(0) = \mathscr{S}^\star$ and $\mathscr{C}(0) = \mathscr{C}$, the familiar subclasses of starlike functions and convex functions in $\mathbb{U}$.

Furthermore, a function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}(\alpha)$, if it satisfies the inequality:

$$\mathrm{Re}\{f'(z)\} > \alpha \quad (0 \le \alpha < 1;\ z \in \mathbb{U}).$$

Note that $\mathscr{R}(\alpha)$ is a subclass of close-to-convex functions of order α $(0 \le \alpha < 1)$ in $\mathbb{U}$. We write $\mathscr{R}(0) = \mathscr{R}$, the familiar class functions whose derivatives have a positive real part in $\mathbb{U}$.

Let $\mathscr{P}$ denote the class of analytic functions of the form

$$\phi(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}) \tag{1.3}$$

such that $\mathrm{Re}\{\phi(z)\} > 0$ in $\mathbb{U}$.

For functions f and g, analytic in the unit disk $\mathbb{U}$, we say the f is said to be subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists an analytic function ω in $\mathbb{U}$ with $\omega(0) = 0, |\omega(z)| \le |z|$ $(z \in \mathbb{U})$ and $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence (see [20]):

$$f(z) \prec g(z)\ (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Following MacGregor [19], we say that f is majorized by g in $\mathbb{U}$ and write

$$f(z) \ll g(z) \quad (z \in \mathbb{U}), \tag{1.4}$$

if there exists a function ψ, analytic in $\mathbb{U}$ such that $|\psi(z)| \le 1$ and

$$f(z) = \psi(z) g(z) \quad (z \in \mathbb{U}). \tag{1.5}$$

For the functions f and g given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \mathbb{U})$$

their Hadamard product (or convolution), denoted by $f \star g$ is defined as

$$(f \star g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g \star f)(z) \quad (z \in \mathbb{U}).$$

We note that $f \star g$ is analytic in $\mathbb{U}$.

For real or complex parameters $a_1, a_2, \ldots, a_p$ and $b_1, b_2, \ldots, b_q$ $(b_j \notin \mathbb{Z}_0^- = \{\ldots, -2, -1, 0\};\ j = 1, 2, \ldots, q)$, the generalized hypergeometric function ${}_pF_q(a_1, a_2, \ldots; b_1, b_2, \ldots, b_q; z)$ is defined by the following infinite series (cf., e.g., [28]):

$${}_pF_q(a_1, a_2, \ldots; b_1, b_2, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (a_q)_n} \frac{z^n}{n!} \tag{1.6}$$

($z \in \mathbb{U}$), where $p \leq q+1, p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$ and $(x)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & (n=0,\ x \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ x(x+1)\cdots(x+n-1), & (n \in \mathbb{N},\ x \in \mathbb{C}). \end{cases}$$

It is easily seen that the radius of convergence ρ of the function ${}_pF_q$ represented by the series (1.6) is

$$\rho = \begin{cases} \infty, & p < q+1, \\ 1, & p = q+1, \\ 0, & p > q+1, \end{cases}$$

so that for $p \leq q+1$, the function ${}_pF_q$ is analytic in $\mathbb{U}$.

By making use of the Hadamard product, Carlson–Shaffer [3] defined the linear operator

$$\mathscr{L}(a,c) : \mathscr{A} \longrightarrow \mathscr{A}$$

in terms of the incomplete beta function φ by

$$\mathscr{L}(a,c)f(z) = \varphi(a,c;z) \star f(z) \quad (f \in \mathscr{A};\ z \in \mathbb{U}), \tag{1.7}$$

where

$$\varphi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad \left(a \in \mathbb{C},\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ z \in \mathbb{U}\right).$$

If $f \in \mathscr{A}$ is given by (1.1), then it follows from (1.7) that

$$\mathscr{L}(a,c)f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \quad (z \in \mathbb{U}) \tag{1.8}$$

and

$$z\left(\mathscr{L}(a,c)f\right)'(z) = a\mathscr{L}(a+1,c)f(z) - (a-1)\mathscr{L}(a,c)f(z) \quad (z \in \mathbb{U}). \tag{1.9}$$

We note that for $f \in \mathscr{A}$

(i) $\mathscr{L}(a,a)f(z) = f(z)$;

(ii) $\mathscr{L}(2,1)f(z) = zf'(z)$;

(iii) $\mathscr{L}(3,1)f(z) = zf'(z) + \frac{1}{2}z^2 f''(z)$;

(iv) $\mathscr{L}(m+1,1)f(z) = \mathscr{D}^m f(z) = \frac{z}{(1-z)^{m+1}} \star f(z)$ ($m \in \mathbb{Z}; m > -1$), the Ruscheweyh derivative operator [26];

(v) $\mathscr{L}(2, 2-\mu)f(z) = \Omega_z^{\mu} f(z)$ ($0 \leq \mu < 1;\ z \in \mathbb{U}$), the well-known Owa–Srivastava fractional differential operator [25]. We also observe that $\Omega_z^0 f(z) = f(z)$ and $\Omega_z^1 f(z) = zf'(z)$.

With the aid of the linear operator $\mathscr{L}(a,c)$, we introduce a subclass of $\mathscr{A}$ as follows:

Definition 1.1. For the fixed parameters A, B $(-1 \leq B < A \leq 1)$, $a > 0$ and $c > 0$, a function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}^{\lambda}(a, c, A, B)$, if it satisfies the following subordination relation:

$$(1.10)\quad (1-\lambda)\frac{\mathscr{L}(a,c)f(z)}{z} + \lambda\frac{\mathscr{L}(a+1,c)f(z)}{z} \prec \frac{1+Az}{1+Bz} \quad (\lambda \geq 0;\ z \in \mathbb{U}).$$

Using the identity (1.9) in (1.10), it follows that

$$\left(1 - \frac{\lambda}{a}\right)\frac{\mathscr{L}(a,c)f(z)}{z} + \frac{\lambda}{a}\left(\mathscr{L}(a,c)f\right)'(z) \prec \frac{1+Az}{1+Bz} \quad (\lambda \geq 0;\ z \in \mathbb{U}).$$

By suitably specializing the parameters a, c, λ, A and B, we obtain the following subclasses of $\mathscr{A}$.

(i) $\mathscr{R}^{0}(a, c, 1-2\alpha, -1) = \mathscr{R}_{a,c}(\alpha)$

$$= \left\{ f \in \mathscr{A} : \operatorname{Re}\left(\frac{\mathscr{L}(a,c)f(z)}{z}\right) > \alpha, 0 \leq \alpha < 1; z \in \mathbb{U} \right\}.$$

(ii) $\mathscr{R}^{2}(2, 2-\mu, \beta(1-2\alpha), -\beta) = \mathscr{R}(\mu, \alpha, \beta)$

$$= \left\{ f \in \mathscr{A} : \left| \frac{\left(\Omega_z^{\mu} f\right)'(z) - 1}{\left(\Omega_z^{\mu} f\right)'(z) + 1 - 2\alpha} \right| < \beta, 0 \leq \alpha < 1, 0 < \beta \leq 1, \right.$$
$$\left. 0 \leq \mu < 1; z \in \mathbb{U} \right\}.$$

We note that $\mathscr{R}(0, \alpha, \beta) = \mathscr{R}(\alpha, \beta)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1)$, the class studied by Juneja and Mogra [10], which in turn give the class considered in [2] for $\beta = 1$.

(iii) $\mathscr{R}^{\lambda}(m+1, 1, 1-2\alpha, -1) = \mathscr{R}_m^{\lambda}(\alpha)$

$$= \left\{ f \in \mathscr{A} : \operatorname{Re}\left((1-\lambda)\frac{\mathscr{D}^m f(z)}{z} + \lambda\frac{\mathscr{D}^{m+1} f(z)}{z}\right) > \alpha, m \in \mathbb{N}_0, \right.$$
$$\left. 0 \leq \alpha < 1; z \in \mathbb{U} \right\}.$$

(iv) $\mathscr{R}^{\lambda}(2, 1, 1-2\alpha, -1) = \mathscr{R}^{\lambda}(\alpha)$

$$= \left\{ f \in \mathscr{A} : \operatorname{Re}\left(f'(z) + \frac{\lambda}{2} z f''(z)\right) > \alpha, 0 \leq \lambda, 0 \leq \alpha < 1; z \in \mathbb{U} \right\}.$$

Next, we define a subclass of $\mathscr{T}$ as follows:

Definition 1.2. For the fixed parameters A, B $(-1 \leq B < A \leq 1,\ 1 \leq B < 0)$, $a > 0$ and $c > 0$, a function $f \in \mathscr{T}$ is said to be in the class $\widetilde{\mathscr{R}}^{\lambda}(a, c, A, B)$, if it satisfies the following subordination relation:

$$(1.11)\qquad (1-\lambda)\frac{\mathscr{L}(a,c)f(z)}{z} + \lambda\frac{\mathscr{L}(a+1,c)f(z)}{z} \prec \frac{1+Az}{1+Bz}$$

$(\lambda \in \mathbb{C},\ \operatorname{Re}(\lambda) \geq 0;\ z \in \mathbb{U})$.

In view of (1.9), it is easily seen that the subordination relation (1.11) is equivalent to

$$\frac{\mathscr{L}(a,c)f(z)}{z}+\frac{\lambda}{a}z\left(\frac{\mathscr{L}(a,c)f(z)}{z}\right)'\prec\frac{1+Az}{1+Bz}\quad(z\in\mathbb{U}).$$

If we set $h(z)=\mathscr{L}(a,c)f(z)/z$, then the above expression further reduces to

$$h(z)+\frac{\lambda}{a}zh'(z)\prec\frac{1+Az}{1+Bz}\quad(\lambda\in\mathbb{C},\mathrm{Re}(\lambda)\geq 0;z\in\mathbb{U}).$$

We write

$$\begin{aligned}&\widetilde{\mathscr{R}}^{\lambda}(1,1,1-2\alpha,-1)=\widetilde{\mathscr{R}}^{\lambda}(\alpha)\\&\qquad=\left\{f\in\mathscr{T}:\mathrm{Re}\left(\frac{f(z)}{z}+\lambda z\left(\frac{f(z)}{z}\right)'\right)>\alpha,z\in\mathbb{U}\right\}\end{aligned}$$

($\lambda\in\mathbb{C}$, $\mathrm{Re}(\lambda)\geq 0$, $0\leq\alpha<1$).

Noonan and Thomas [23] defined the q-th Hankel determinant of the function $f\in\mathscr{A}$ given by (1.1) as

$$H_q(n)=\begin{vmatrix}a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}\quad(a_1=1;n,q\in\mathbb{N}).$$

This determinant has been studied by several authors with the subject of inquiry ranging from the rate of growth of $H_q(n)$ (as $n\to\infty$) [24] to the determination of precise bounds with specific values of n and q for certain subclasses of analytic functions in the unit disc $\mathbb{U}$.

For $n=1,q=2=1$ and $n=q=2$, the Hankel determinant simplifies to

$$H_2(1)=|a_3-a_2^2|\quad\text{and}\quad H_2(2)=|a_2a_4-a_3^2|.$$

We refer to $H_2(2)$ as the second Hankel determinant. It is known [4] that if f given by (1.1) is analytic and univalent in $\mathbb{U}$, then the sharp inequality $H_2(1)=|a_3-a_2^2|\leq 1$ holds. For a family $\mathscr{F}$ of functions in $\mathscr{A}$ of the form (1.1), the more general problem of finding the sharp upper bounds for the functionals $|a_3-\mu\,a_2^2|$ ($\mu\in\mathbb{R}/\mathbb{C}$) is popularly known as Fekete–Szegö problem for the class $\mathscr{F}$. The Fekete–Szegö problem for the known classes of univalent functions, that is, starlike functions, convex functions and close-to-convex functions has been completely settled [5, 11, 12, 13]. Recently, Janteng et al. [8, 9] have obtained the sharp upper bounds to the second Hankel determinant $H_2(2)$ for the family $\mathscr{R}$. For initial work on the class $\mathscr{R}$, one may refer to the paper by MacGregor [17].

A majorization properties for the class of starlike functions of complex order γ and the class of convex functions of complex order γ ($\gamma\in\mathbb{C}^*$) has been investigated by Altintaş et al. [1] and MacGregor [19] has also studied the same problem for the classes $\mathscr{S}^\star$ and $\mathscr{C}$, respectively. Recently, Goyal and Goswami [6], and Goyal et al. [7] generalized these results for different function classes.

In the present article, by following the techniques devised by Libera and Złotkiewicz [14, 15], we solve the Fekete–Szegö problem and also determine the sharp upper bound to the second Hankel determinant for the class $\mathscr{R}^\lambda(a,c,A,B)$. We also investigate several majorization properties for certain subclasses of analytic functions in the unit disk $\mathbb{U}$. Relevant connections of the results presented here with those obtained in earlier works are also mentioned.

To establish our main results, we shall need the following lemmas.

Lemma 1.1 ([4]). *Let the function ϕ, given by* (1.3) *be a member of the class $\mathscr{P}$. Then*

$$|p_k| \le 2 \quad (k \ge 1)$$

and the estimate is sharp for the function $\phi(z) = (1+z)/(1-z)$, $z \in \mathbb{U}$.

Lemma 1.2 ([16]). *If the function ϕ, given by* (1.3) *belongs to the class $\mathscr{P}$, then for any $\gamma \in \mathbb{C}$*

$$\left|p_2 - \gamma\, p_1^2\right| \le 2\max\{1, |2\gamma - 1|\}$$

and the result is sharp for the functions given by

$$\phi(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad \phi(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{U}).$$

Lemma 1.3 ([15], see also [14]). *If the function ϕ, given by* (1.3) *belongs to the class $\mathscr{P}$, then*

$$p_2 = \frac{1}{2}\left\{p_1^2 + (4-p_1^2)x\right\}$$

and

$$p_3 = \frac{1}{4}\left\{p_1^3 + 2(4-p_1^2)p_1 x - (4-p_1^2)p_1 x^2 + 2(4-p_1^2)(1-|x|^2)z\right\}$$

for some complex numbers x, z satisfying $|x| \le 1$ and $|z| \le 1$.

2. Hankel determinant for the class $\mathscr{R}^\lambda(a,c,A,B)$. Unless otherwise mentioned, we assume throughout the sequel that

$$a > 0,\ c > 0,\ \lambda \ge 0 \ \text{ and } \ -1 \le B < A \le 1.$$

Now, we determine the sharp upper bound for the functional $|a_3 - \gamma a_2^2|$ $(\gamma \in \mathbb{R})$ for functions belonging to the class $\mathscr{R}^\lambda(a,c,A,B)$.

Theorem 2.1. *If $\gamma \in \mathbb{R}$ and the function f, given by* (1.1) *belongs to the class $\mathscr{R}^{\lambda}(a,c,A,B)$, then*

$$\left|a_3-\gamma a_2^2\right| \leq \begin{cases} -\frac{(c)_2(A-B)}{(a+1)(a+2\lambda)}\left\{B+\frac{(a+1)(a+2\lambda)c(A-B)\gamma}{(a+\lambda)^2(c+1)}\right\}, & \gamma<\rho_1, \\ \frac{(c)_2(A-B)}{(a+1)(a+2\lambda)}, & \rho_1 \leq \gamma \leq \rho_2, \\ \frac{(c)_2(A-B)}{(a+1)(a+2\lambda)}\left\{B+\frac{(a+1)(a+2\lambda)c(A-B)\gamma}{(a+\lambda)^2(c+1)}\right\}, & \gamma>\rho_2, \end{cases} \tag{2.1}$$

where

$$\rho_1=-\frac{(1+B)(a+\lambda)^2(c+1)}{(a+1)(a+2\lambda)c(A-B)} \quad \text{and} \quad \rho_2=\frac{(1-B)(a+\lambda)^2(c+1)}{(a+1)(a+2\lambda)c(A-B)}.$$

The estimate in (2.1) *is sharp.*

Proof. From (1.10), we have

$$(1-\lambda)\frac{\mathscr{L}(a,c)f(z)}{z}+\lambda\frac{\mathscr{L}(a+1,c)f(z)}{z}=\frac{1-A+(1+A)\phi(z)}{1-B+(1+B)\phi(z)} \tag{2.2}$$

($z \in \mathbb{U}$), where the function ϕ, given by (1.3) belongs to the class $\mathscr{P}$. Writing the series expansion of $\mathscr{L}(a,c)f(z)/z, \mathscr{L}(a+1,c)f(z), \phi(z)$ from (1.8) and (1.3) in (2.2), and comparing the like powers of z in the resulting equation, we deduce that

$$a_2=\frac{c(A-B)}{2(a+\lambda)}p_1 \tag{2.3}$$

$$a_3=\frac{c(c+1)(A-B)}{2(a+1)(a+2\lambda)}\left\{p_2-\frac{1}{2}(1+B)p_1^2\right\} \tag{2.4}$$

and

$$a_4=\frac{c(c+1)(c+2)(A-B)}{2(a+1)(a+2)(a+3)}\left\{p_3-(1+B)p_1p_2+\frac{1}{4}(1+B)^2p_1^3\right\}. \tag{2.5}$$

Using (2.3) and (2.4), we obtain

$$\left|a_3-\gamma a_2^2\right|=\frac{(c)_2(A-B)}{(a+1)(a+2\lambda)}\left|p_2-\left\{\frac{(a+1)(a+2\lambda)c(A-B)}{2(a+\lambda)^2(c+1)}\gamma+B+1\right\}p_1^2\right|$$

and with the aid of Lemma 1.2, the above expression yields

$$\begin{aligned} &\left|a_3-\gamma a_2^2\right| \\ &\quad \leq \frac{2(c)_2(A-B)}{(a+1)(a+2\lambda)}\max\left\{1,\left|\frac{(a+1)(a+2\lambda)c(A-B)}{(a+\lambda)^2(c+1)}\gamma+B\right|\right\}. \end{aligned} \tag{2.6}$$

If $\gamma<\rho_1$, then

$$\frac{(a+1)(a+2\lambda)c(A-B)}{(a+\lambda)^2(c+1)}\gamma+B<-1,$$

which in view of (2.6) implies the first case of the estimate in (2.1). In the case $\rho_1 \leq \gamma \leq \rho_2$, we obtain

$$\left|\frac{(a+1)(a+2\lambda)c(A-B)}{(a+\lambda)^2(c+1)}\gamma + B\right| \leq 1.$$

Thus, from (2.6), we get the second case of the estimate in (2.1). Finally, for $\gamma > \rho_2$, we deduce that

$$\left|\frac{(a+1)(a+2\lambda)c(A-B)}{(a+\lambda)^2(c+1)}\gamma + B\right| > 1,$$

which again with the aid of (2.6) gives the third case of the estimate in (2.1).

It is easily seen that the estimate for the first and third cases in (2.1) are sharp for the function f, defined in $\mathbb{U}$ by

$$f(z) = \begin{cases} \varphi(c,a;z) \star \frac{z(1+Az)}{1+Bz}, & \lambda = 0, \\ z\,{}_3F_2\left(1, \frac{a}{\lambda}, c; a, 1+\frac{a}{\lambda}; z\right) \star \frac{z(1+Az)}{1+Bz}, & \lambda > 0. \end{cases}$$

The estimate for the second case in (2.1) is sharp for the function f, defined in $\mathbb{U}$ by

$$f(z) = \begin{cases} \varphi(c,a;z) \star \frac{z(1+Az^2)}{1+Bz^2}, & \lambda = 0, \\ z\,{}_3F_2\left(1, \frac{a}{\lambda}, c; a, 1+\frac{a}{\lambda}; z\right) \star \frac{z(1+Az^2)}{1+Bz^2}, & \lambda > 0, \end{cases} \tag{2.7}$$

where the function ${}_3F_2$ is defined by (1.6). □

Setting $a = 2$, $c = 2-\mu$ $(0 \leq \mu < 1)$, $A = \beta(1-2\alpha)$, $B = -\beta$ $(0 \leq \alpha < 1,$ $0 < \beta \leq 1)$ and $\lambda = 2$ in Theorem 2.1, we obtain the following result.

Corollary 2.1. *If $\gamma \in \mathbb{R}$ and the function f, given by* (1.1) *belongs to the class $\mathscr{R}(\mu,\alpha,\beta)$, then*

$$\left|a_3 - \gamma a_2^2\right| \leq \begin{cases} \frac{\beta(1-\alpha)(2-\mu)(3-\mu)}{9}\left\{\beta - \frac{9\beta(1-\alpha)(2-\mu)\gamma}{4(3-\mu)}\right\}, & \gamma < \sigma_1, \\ \frac{\beta(1-\alpha)(2-\mu)(3-\mu)}{9}, & \sigma_1 \leq \gamma \leq \sigma_2, \\ \frac{\beta(1-\alpha)(2-\mu)(3-\mu)}{9}\left\{\frac{9\beta(1-\alpha)(2-\mu)\gamma}{4(3-\mu)} - \beta\right\}, & \gamma > \sigma_2, \end{cases}$$

where

$$\sigma_1 = -\frac{4(1+\beta)(3-\mu)}{9\beta(1-\alpha)(2-\mu)} \quad \textit{and} \quad \sigma_2 = \frac{4(1-\beta)(3-\mu)}{9\beta(1-\alpha)(2-\mu)}.$$

The estimate is sharp for the functions f, defined in $\mathbb{U}$ by

$$f(z) = z\,{}_3F_2(1,1,2-\mu;2,2;z) \star \frac{z\{1+\beta(1-2\alpha)z\}}{1-\beta z}$$

and

$$f(z) = z\,{}_3F_2(1,1,2-\mu;2,2;z) \star \frac{z\{1+\beta(1-2\alpha)z^2\}}{1-\beta z^2}.$$

Letting $a = 2$, $c = 1$, $A = 1 - 2\alpha$ $(0 \le \alpha < 1)$ and $B = -1$ in Theorem 1, we get

Corollary 2.2. *If $\gamma \in \mathbb{R}$ and the function f, given by* (1.1) *belongs to the class $\mathscr{R}^\lambda(\alpha)$, then*

$$\left|a_3 - \gamma a_2^2\right| \le \begin{cases} \frac{2(1-\alpha)}{3(1+\lambda)}\left\{1 - \frac{6(1+\lambda)(1-\alpha)\gamma}{(2+\lambda)^2}\right\}, & \gamma < 0, \\ \frac{2(1-\alpha)}{3(1+\lambda)}, & 0 \le \gamma \le \frac{(2+\lambda)^2}{3(1+\lambda)(1-\alpha)}, \\ \frac{2(1-\alpha)}{3(1+\lambda)}\left\{\frac{6(1+\lambda)(1-\alpha)\gamma}{(2+\lambda)^2} - 1\right\}, & \gamma > \frac{(2+\lambda)^2}{3(1+\lambda)(1-\alpha)}. \end{cases}$$

The estimate is sharp for the functions f, defined in $\mathbb{U}$ by

$$f(z) = \begin{cases} \varphi(1, 2; z) \star \frac{z\{1+(1-2\alpha)z\}}{1-z}, & \lambda = 0, \\ z\,{}_3F_2(1, 1, 2-\mu; 2, 2; z) \star \frac{z\{1+(1-2\alpha)z\}}{1-z}, & \lambda > 0 \end{cases}$$

and

$$f(z) = \begin{cases} \varphi(1, 2; z) \star \frac{z\{1+(1-2\alpha)z\}}{1-z}, & \lambda = 0, \\ z\,{}_3F_2(1, 1, 2-\mu; 2, 2; z) \star \frac{z\{1+(1-2\alpha)z^2\}}{1-z^2}, & \lambda > 0. \end{cases}$$

In the following theorem, we find the sharp upper bound to the second Hankel determinant for the class $\mathscr{R}^\lambda(a, c, A, B)$.

Theorem 2.2. *Let $a \ge c > 0, \lambda \ge 0, -1 \le B < A \le 1$ and*

$$2(1+|B|)(a+2)(c+1)(a+\lambda)(a+3\lambda) \ge (1+2|B|)(a+1)(c+2)(a+2\lambda)^2. \tag{2.8}$$

If the function f, given by (1.1) *belongs to the class $\mathscr{R}^\lambda(a, c, A, B)$, then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{(c)_2^2(A-B)^2}{(a+1)^2(a+2\lambda)^2}. \tag{2.9}$$

The estimate in (2.9) *is sharp.*

Proof. Assuming that f, given by (1.1) belongs to the class $\mathscr{R}^\lambda(a, c, A, B)$ and using (2.3), (2.4) and (2.5), we deduce that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &= \frac{c^2(c+1)(A-B)^2}{4(a+1)(a+\lambda)(a+3\lambda)} \\ &\times \left|K_1p_1p_3 - (1+B)(K_1 - K_2)p_1^2p_2 + \frac{(1+B)^2(K_1-K_2)}{4}p_1^4\right|, \end{aligned} \tag{2.10}$$

where

$$K_1 = \frac{c+2}{a+2} \quad \text{and} \quad K_2 = \frac{(c+1)(a+\lambda)(a+3\lambda)}{(a+1)(a+2\lambda)^2}.$$

Since the functions $\phi(z)$ and $\phi(e^{i\theta}z)$ $(\theta \in \mathbb{R})$, defined by (1.3) are in the class $\mathscr{P}$ simultaneously, we assume without loss of generality that $p_1 > 0$.

For convenience, we write $p_1 = p$ $(0 \le p \le 2)$. Now, by using Lemma 1.3 in (2.10), we get

$$
\begin{aligned}
&|a_2a_4 - a_3^2| \\
&\quad = \frac{c^2(c+1)(A-B)^2}{4(a+1)(a+\lambda)(a+3\lambda)} \left| \left\{ \frac{K_1}{4}p^4 + \frac{K_1}{2}(4-p^2)p^2x \right.\right. \\
&\qquad \left. - \frac{K_1}{4}(4-p^2)p^2x^2 + \frac{K_1}{2}(4-p^2)p(1-|x|^2)z \right\} \\
&\qquad - \left\{ \frac{(1+B)(K_1-K_2)}{2}p^4 + \frac{(1+B)(K_1-K_2)}{2}(4-p^2)p^2 \right\} \\
&\qquad - \left\{ \frac{K_2}{4}p^4 + \frac{K_2}{2}(4-p^2)p^2x + \frac{K_2}{4}(4-p^2)^2x^2 \right\} \\
&\qquad \left. + \frac{(1+B)^2(K_1-K_2)}{4}p^4 \right| \\
&\quad = \frac{c^2(c+1)(A-B)^2}{4(a+1)(a+\lambda)(a+3\lambda)} \left| \frac{B^2(K_1-K_2)}{4}p^4 - \frac{B(K_1-K_2)}{2}(4-p^2)p^2x \right. \\
&\qquad \left. - \frac{1}{4}(4-p^2)\left\{K_1p^2 + K_2(4-p^2)\right\}x^2 + \frac{K_1}{2}(4-p^2)p(1-|x|^2)z \right|
\end{aligned} \tag{2.11}
$$

for some complex numbers x $(|x| \le 1)$ and z $(|z| \le 1)$. Applying the triangle inequality in (2.11) and upon replacing $|x|$ by y in the resulting expression, we get

$$
\begin{aligned}
\left|a_2a_4 - a_3^2\right| \le{}& \frac{c^2(c+1)(A-B)^2}{4(a+1)(a+\lambda)(a+3\lambda)} \left\{ \frac{|B|^2(K_1-K_2)}{4}p^4 \right. \\
&+ \frac{|B|(K_1-K_2)}{2}(4-p^2)p^2y \\
&+ \frac{1}{4}(4-p^2)\left\{K_1p^2 + K_2(4-p^2) - 2K_1p\right\}y^2 \\
&\left. + \frac{K_1}{2}(4-p^2)p \right\} \\
=:{}& \mathcal{G}(p,y)
\end{aligned} \tag{2.12}
$$

$(0 \le p \le 2,\ 0 \le y \le 1)$. We next maximize the function $\mathcal{G}(p,y)$ on the closed rectangle $[0,2] \times [0,1]$. Since

$$2K_2 > K_1 > K_2 > 0,$$

we have

$$
\begin{aligned}
\frac{\partial \mathcal{G}}{\partial y} ={}& \frac{c^2(c+1)(A-B)^2}{8(a+1)(a+\lambda)(a+3\lambda)}(4-p^2) \\
&\times \left[(K_1-K_2)|B|p^2 + (2-p)\left\{(2+p)K_2 - K_1p\right\}y\right] > 0
\end{aligned}
$$

for $0 < p < 2$ and $0 < y < 1$. Thus, $\mathcal{G}(p,y)$ cannot have a maximum in the interior on the closed rectangle $[0,2] \times [0,1]$. Therefore, for fixed $p \in [0,2]$

$$\max_{0 \le y \le 1} \mathcal{G}(p,y) = \mathcal{G}(p,1) =: \mathcal{F}(p),$$

where

$$\begin{aligned}\mathcal{F}(p) = \frac{c^2(c+1)(A-B)^2}{8(a+1)(a+\lambda)(a+3\lambda)} &\left[\frac{|B|^2(K_1-K_2)}{4}p^4\right. \\ &+ \left.\frac{|B|(K_1-K_2)}{2}(4-p^2)p^2 + \frac{1}{4}(4-p^2)\left\{K_1p^2 + K_2(4-p^2)\right\}\right].\end{aligned}$$

A routine calculation yields

$$\mathcal{F}'(p) = p\big\{(|B|^2 - 2|B| - 1)(K_1-K_2)p^2 + 2(1+2|B|)K_1 - 4(1+|B|)K_2\big\}.$$

Thus, $\mathcal{F}'(p) = 0$ implies that either $p = 0$ or

$$p^2 = -\frac{2\left\{2(1+|B|)K_2 - (1+2|B|)K_1\right\}}{\{2-(1-|B|^2)\}(K_1-K_2)\}} < 0,$$

which is not true. We, further observe that

$$\mathcal{F}''(0) = 2\left\{(1+2|B|)K_1 - 2(1+|B|)K_2\right\} < 0$$

by (2.8). Since $\mathcal{F}(2) < \mathcal{F}(0)$, $\max_{0 \le p \le 2} \mathcal{F}(p)$ occurs at $p = 0$. Thus, the upper bound of (2.12) corresponds to $p = 0$ and $y = 1$, from which we get the estimate in (2.9).

It is easily seen that the estimate (2.9) is sharp for the function f, given by (2.7) and thus the proof of Theorem 2.2 is completed. □

Setting $\lambda = 0$, $A = 1 - 2\alpha$ $(0 \le \alpha < 1)$ and $B = -1$ in Theorem 2.2, we get the following result obtained by Mishra and Kund [21].

Corollary 2.3. *If $a \ge c > 0$, $ac - 2a + 5c + 2 \ge 0$ and the function f, given by* (1.1) *belongs to the class $\mathscr{R}_{a,c}(\alpha)$, then*

$$\left|a_2a_4 - a_3^2\right| \le \left\{\frac{2(c)_2(1-\alpha)}{(a)_2}\right\}^2.$$

The estimate is sharp for the function f, defined by

$$f(z) = \varphi(c,a;z) \star \frac{z\{1+(1-2\alpha)z^2\}}{1-z^2} \quad (0 \le \alpha < 1;\ z \in \mathbb{U}).$$

Letting $a = 2$, $c = 2 - \mu$, $A = \beta(1-2\alpha)$, $B = -\beta$ and $\lambda = 2$ in Theorem 2.2, we obtain

Corollary 2.4. *If the function f, given by* (1.1) *belongs to the class $\mathscr{R}(\mu,\alpha,\beta)$, then*

$$\left|a_2a_4 - a_3^2\right| \le \frac{\{\beta(1-\alpha)(2-\mu)(3-\mu)\}^2}{9} \quad (0 \le \mu < 1, 0 \le \alpha < 1, 0 < \beta \le 1)$$

and the estimate is sharp for the function f, defined by

$$f(z) = z\,{}_3F_2(1,1,2-\mu;2,2;z) \star \frac{z\{1+\beta(1-2\alpha)z^2\}}{1-\beta z^2} \quad (z \in \mathbb{U}).$$

Taking $a = 2$, $c = 1$, $A = 1-2\alpha$ and $B = -1$ in Theorem 2.2, we get the following result, which in turn yields the corresponding work of Mishra and Kund [21] for $\lambda = 0$, and the work of Janteng et al. [8] for $\lambda = \alpha = 0$.

Corollary 2.5. *If the function f, given by* (1.1) *belongs to the class $\mathscr{R}^\lambda(\alpha)$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4(1-\alpha)^2}{9(1+\lambda)^2} \qquad \left(0 \leq \alpha < 1, 0 \leq \lambda \leq \frac{5+2\sqrt{10}}{3}\right)$$

and the estimate is sharp for the function f, defined in $\mathbb{U}$ by

$$f(z) = \begin{cases} \varphi(1,2;z) \star \frac{z\{1+(1-2\alpha)z^2\}}{1-z^2}, & \lambda = 0 \\ z\,{}_3F_2\left(1,1,\frac{2}{\lambda};2,1+\frac{2}{\lambda};z\right) \star \frac{z\{1+(1-2\alpha)z^2\}}{1-z^2}, & \lambda > 0. \end{cases}$$

3. Majorization properties. We prove the following lemmas, which will be used in our investigation of majorization properties for the class $\widetilde{\mathscr{R}}^\lambda(a,c,A,B)$.

Lemma 3.1. *If $a \geq c > 0$ and the function g, given by* (1.2) *belongs to the class $\widetilde{\mathscr{R}}^\lambda(a,c,A,B)$, then*

$$\sum_{n=1}^{\infty} b_{n+1} \leq \frac{c(A-B)}{\{a+Re(\lambda)\}(1-B)}. \tag{3.1}$$

Proof. It follows from (1.11) that

$$\mathrm{Re}\left\{(1-\lambda)\frac{\mathscr{L}(a,c)f(z)}{z} + \lambda\frac{\mathscr{L}(a+1,c)f(z)}{z}\right\} > \frac{1-A}{1-B} \quad (z \in \mathbb{U})$$

which upon substituting the series expansion of $\mathscr{L}(a,c)f(z)/z$ and $\mathscr{L}(a+1,c)f(z)/$ gives

$$\mathrm{Re}\left\{1 - \sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}\left(1+\frac{\lambda n}{a}\right)b_{n+1}z^n\right\} > \frac{1-A}{1-B} \quad (z \in \mathbb{U}).$$

Letting $z \to 1^-$ through real values in the above expression, we find that

$$1 - \sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}\left\{1+\frac{\mathrm{Re}(\lambda)n}{a}\right\}b_{n+1} > \frac{1-A}{1-B}.$$

Since $a \geq c > 0$, $b_{n+1} \geq 0$ and $\mathrm{Re}(\lambda) \geq 0$, the above inequality implies that

$$\frac{a}{c}\left\{1+\frac{\mathrm{Re}(\lambda)}{a}\right\}\sum_{n=1}^{\infty} b_{n+1} \leq \sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}\left\{1+\frac{\mathrm{Re}(\lambda)n}{a}\right\}b_{n+1} \leq \frac{A-B}{1-B}.$$

This completes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Under the hypothesis of Lemma 3.1, we have for* $|z| = r < 1$

$$\begin{aligned} 1 - \frac{c(A-B)r}{\{a + Re(\lambda)\}(1-B)} &\leq Re\left(\frac{g(z)}{z}\right) \\ &\leq \left|\frac{g(z)}{z}\right| \leq 1 - \frac{c(A-B)r}{\{a + Re(\lambda)\}(1-B)}. \end{aligned} \tag{3.2}$$

Proof. Since

$$\left|\frac{g(z)}{z}\right| \leq 1 + \left(\sum_{n=1}^{\infty} b_{n+1}\right)|z| \quad (z \in \mathbb{U}),$$

Lemma 3.1 leads to

$$\left|\frac{g(z)}{z}\right| \leq 1 + \frac{c(A-B)r}{\{a + \mathrm{Re}(\lambda)\}(1-B)} \quad (|z| = r < 1).$$

Similarly, we have

$$\begin{aligned} \mathrm{Re}\left(\frac{g(z)}{z}\right) &= 1 - \mathrm{Re}\left(\sum_{n=1}^{\infty} b_{n+1} z^n\right) \geq 1 - \left|\sum_{n=1}^{\infty} b_{n+1} z^n\right| \\ &\geq 1 - \left(\sum_{n=1}^{\infty} b_{n+1}\right)|z| \geq 1 - \frac{c(A-B)r}{\{a + \mathrm{Re}(\lambda)\}(1-B)} \quad (|z| = r < 1) \end{aligned}$$

and the proof of Lemma 3.2 is completed. □

Now, we prove

Theorem 3.1. *Let the function* g *be in the class* $\mathscr{T}$. *If* $a \geq c > 0$, *the function* $h \in \mathscr{T}$ *satisfies*

$$\frac{z^2 (\mathscr{L}(a,c)h)'(z)}{\mathscr{L}(a,c)h(z)} \in \widetilde{\mathscr{R}}^{\lambda}(a,c,A,B) \tag{3.3}$$

and $\mathscr{L}(a,c)g \ll \mathscr{L}(a,c)h$ *in* $\mathbb{U}$, *then*

$$\left|(\mathscr{L}(a,c)g)'(z)\right| \leq \left|(\mathscr{L}(a,c)h)'(z)\right| \quad (|z| \leq r(\lambda,a,c,A,B)), \tag{3.4}$$

where $r(\lambda,a,c,A,B)$ *is the root of the cubic equation*

$$\begin{aligned} c(A-B)r^3 &- \{a + Re(\lambda)\}(1-B)r^2 \\ &- [2\{a + Re(\lambda)\}(1-B) + c(A-B)]\, r \\ &+ \{a + Re(\lambda)\}(1-B) = 0 \end{aligned} \tag{3.5}$$

in $(0,1)$.

Proof. From (3.3), by using Lemma 3.2, we get for $|z| = r < 1$

$$|\mathcal{L}(a,c)h(z)| \leq \frac{\{a + \mathrm{Re}(\lambda)\}(1-B)r}{\{a + \mathrm{Re}(\lambda)\}(1-B) - c(A-B)r} \left|(\mathcal{L}(a,c)h)'(z)\right|. \tag{3.6}$$

Since $\mathscr{L}(a,c)g(z) \ll \mathscr{L}(a,c)h(z)$ in $\mathbb{U}$, we have by (1.5)

$$\mathscr{L}(a,c)g(z) = \psi(z)\mathscr{L}(a,c)h(z) \quad (z \in \mathbb{U}),$$

where the function ψ is analytic in $\mathbb{U}$ and satisfies $|\psi(z)| \leq 1$ in $\mathbb{U}$, so that

$$(3.7)\quad \left(\mathscr{L}(a,c)g\right)'(z) = \psi(z)\left(\mathscr{L}(a,c)h\right)'(z) + \psi'(z)\mathscr{L}(a,c)h(z) \quad (z \in \mathbb{U}).$$

Using the following estimate [22]

$$(3.8)\qquad \left|\psi'(z)\right| \leq \frac{1-|\psi(z)|^2}{1-|z|^2} \quad (z \in \mathbb{U})$$

followed by (3.6) in (3.7), we obtain

$$\begin{aligned}&\left|\left(\mathscr{L}(a,c)g\right)'(z)\right| \\ &\leq \left[|\psi(z) + \frac{1-|\psi(z)|^2}{1-|z|^2}\frac{\{a+\mathrm{Re}(\lambda)\}(1-B)r}{\{a+\mathrm{Re}(\lambda)\}(1-B)-c(A-B)r}\right]\left|\left(\mathscr{L}(a,c)h\right)'(z)\right|\end{aligned}$$

which upon setting $|\psi(z)| = x$ $(0 \leq x \leq 1)$ yields the inequality

$$(3.9)\quad \begin{aligned}&\left|\left(\mathscr{L}(a,c)g\right)'(z)\right| \\ &\leq \left\{\frac{\Psi(x)}{(1-r^2)\left\{(a+\mathrm{Re}(\lambda))(1-B)-c(A-B)r\right\}}\right\}\left|\left(\mathscr{L}(a,c)h\right)'(z)\right|,\end{aligned}$$

where

$$\begin{aligned}\Psi(x) = &\{a+\mathrm{Re}(\lambda)\}(1-B)rx^2 \\ &+ (1-r^2)\left[\{a+\mathrm{Re}(\lambda)\}(1-B)-c(A-B)r\right]x \\ &+ \{a+\mathrm{Re}(\lambda)\}(1-B)r.\end{aligned}$$

The function Ψ attains its maximum value at $x = 1$ with $r = r(\lambda, a, c, A, B)$, the root of the equation (3.5) contained in $(0, 1)$. Furthermore, if $0 \leq y \leq r(\lambda, a, c, A, B)$, then the function

$$\begin{aligned}\Theta(x) = &-\{a+\mathrm{Re}(\lambda)\}(1-B)yx^2 \\ &+ (1-y^2)\left[\{a+\mathrm{Re}(\lambda)\}(1-B)-c(A-B)y\right]x \\ &+ \{a+\mathrm{Re}(\lambda)\}(1-B)y\end{aligned}$$

increases in the interval $0 \leq x \leq 1$, so that

$$\Theta(x) \leq \Theta(1) = (1-y^2)\left[\{a+\mathrm{Re}(\lambda)\}(1-B)-c(A-B)y\right].$$

Thus, by substituting $x = 1$ in (3.9), we conclude that the inequality in (3.4) holds true for $|z| \leq r(\lambda, a, c, A, B)$, where $r(\lambda, a, c, A, B)$ is given by (3.5). This completes the proof of Theorem 3.1. □

For $a = c = 1$, $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$ and $B = -1$, Theorem 3.1 gives the following result.

Corollary 3.1. *If the function* $g \in \mathscr{T}$ *and* $g \ll h$ *in* $\mathbb{U}$, *where* $z^2h'(z)/h(z) \in \widetilde{\mathscr{R}}^{\lambda}(\alpha)$, *then*

$$\left|g'(z)\right| \leq \left|h'(z)\right| \quad (|z| \leq r(\lambda, \alpha)),$$

where $r(\lambda, \alpha)$ *is the root of the cubic equation*

$$(1-\alpha)r^3 - \{1 + Re(\lambda)\}r^2 - \{3 - \alpha + 2Re(\lambda)\}r + 1 + Re(\lambda) = 0$$

in $(0, 1)$.

In the special case $\lambda = 0$, Corollary 3.1 simplifies to the following result.

Corollary 3.2. *Let the function* g *be in the class* $\mathscr{T}$. *If the function* $h \in \mathscr{S}^\star(\alpha) \cap \mathscr{T}$ *and* $g \ll h$ *in* $\mathbb{U}$, *then*

$$|g'(z)| \leq |h'(z)| \quad (|z| \leq r(\alpha)),$$

where $r(\alpha)$ *is the root of the cubic equation* $(1-\alpha)r^3 - r^2 - (3-\alpha)r + 1 = 0$ *in* $(0, 1)$.

With the aid of the following inclusion relation [27, Theorem 7]

$$\mathscr{C}(\alpha) \cap \mathscr{T} \subset \mathscr{S}^\star\left(\frac{2}{3-\alpha}\right) \cap \mathscr{T} \quad (0 \leq \alpha < 1),$$

we get the following result from Corollary 3.2.

Corollary 3.3. *Let the function* g *be in the class* $\mathscr{T}$. *If the function* $h \in \mathscr{C}(\alpha) \cap \mathscr{T}$ *and* $g \ll h$ *in* $\mathbb{U}$, *then*

$$|g'(z)| \leq |h'(z)| \quad (|z| \leq \widetilde{r}(\alpha)),$$

where $\widetilde{r}(\alpha)$ *is the root of the cubic equation* $(1-\alpha)r^3 - (3-\alpha)r^2 - (7 - 3\alpha)r + (3-\alpha) = 0$ *in* $(0, 1)$.

Finally, we prove

Theorem 3.2. *Let the function* f *be in the class* $\mathscr{A}$. *If the function* $g \in \mathscr{A}$ *satisfies the subordination condition:*

$$\frac{z\left(\mathscr{L}(a,c)g\right)'(z)}{\mathscr{L}(a,c)g(z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1;\ z \in \mathbb{U}) \tag{3.10}$$

and $\mathscr{L}(a,c)f \ll \mathscr{L}(a,c)g$ *in* $\mathbb{U}$, *then*

$$|\mathscr{L}(a+1,c)f(z)| \leq |\mathscr{L}(a+1,c)f(z)| \quad (|z| \leq r(a,A,B)), \tag{3.11}$$

where $r(a, A, B)$ *is the root of the cubic equation*

$$|A + (a-1)B|r^3 - (a + 2|B|)r^2 - (|A + (a-1)B| + 2)r + a = 0 \tag{3.12}$$

in $(0, 1)$.

Proof. From the definition of subordination, it follows from (3.10) that

$$\frac{z\left(\mathscr{L}(a,c)g\right)'(z)}{\mathscr{L}(a,c)g(z)} = \frac{1+A\omega(z)}{1+B\omega(z)} \quad (z \in \mathbb{U}), \tag{3.13}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. Now, making use of the the identity (1.9) for the function g in (3.13), we deduce that

$$\begin{aligned} |\mathscr{L}(a,c)g(z)| &\le \frac{a(1+|B||\omega(z)|)}{a-|A+(a-1)B||\omega(z)|}\,|\mathscr{L}(a+1,c)g(z)| \\ &\le \frac{a(1+|B||z|)}{a-|A+(a-1)B||z|)}\,|\mathscr{L}(a+1,c)g(z)| \quad (z\in\mathbb{U}). \end{aligned} \tag{3.14}$$

Since $\mathscr{L}(a,c)f$ is majorized by $\mathscr{L}(a,c)g$ in $\mathbb{U}$, we have

$$\mathscr{L}(a,c)f(z) = \psi(z)\mathscr{L}(a,c)g(z) \quad (z\in\mathbb{U}),$$

where ψ is analytic in U and satisfies $|\psi(z)| \le 1$ in U. Differentiating the above expression with respect to z, using the identity (1.9) for both the functions f and g in the resulting equation, we obtain

$$a\,\mathscr{L}(a+1,c)f(z) = a\,\psi(z)\mathscr{L}(a+1,c)g(z) + z\psi'(z)\mathscr{L}(a,c)g(z) \tag{3.15}$$

$(z\in\mathbb{U})$. Using the estimate (3.8) and (3.14) in (3.15), we get

$$\begin{aligned} &|\mathscr{L}(a+1,c)f(z)| \\ &\le \left\{|\psi(z)| + \frac{1-|\psi(z)|^2}{1-|z|^2}\frac{(1+|B||z|)|z|}{a-|A+(a-1)B||z|}\right\}|\mathscr{L}(a+1,c)g(z)| \end{aligned}$$

which upon setting $|z| = r$ and $|\psi(z)| = x$ $(0\le x\le 1)$ yields the inequality

$$|\mathscr{L}(a+1,c)f(z)| \le \left[\frac{\Psi(x)}{(1-r^2)\{a-|A+(a-1)B|r\}}\right]|\mathscr{L}(a+1,c)g(z)|,$$

where

$$\Psi(x) = -r(1+|B|r)x^2 + (1-r^2)\{a-|A+(a-1)B|r\}x + (1+|B|r)r$$

which takes on its maximum value at $x = 1$ with $r = r(a,A,B)$, where $r(a,A,B)$ is the root of the equation (3.12) in $(0,1)$.

The remaining part of the proof of Theorem 3.2 is much akin to that of Theorem 3.1, and so we omit the details. □

Letting $a = c = 1$, $A = \beta(1-2\alpha)$ and $B = -\beta$ in Theorem 3.2, we get

Corollary 3.4. *Let the function f be in the class $\mathscr{A}$. If the function $g\in\mathscr{A}$ satisfies*

$$\left|\frac{zg'(z)/g(z)-1}{zg'(z)/g(z)+1-2\alpha}\right| < \beta \quad (0\le\alpha<1,\ 0<\beta\le 1;\ z\in\mathbb{U})$$

and $f \ll g$ in $\mathbb{U}$, then

$$|f'(z)| \le |g'(z)| \quad (|z|\le r(\alpha,\beta)),$$

where

$$r(\alpha,\beta) = \begin{cases} \frac{3+\beta|1-2\alpha|-\sqrt{\beta^2|1-2\alpha|^2+2\beta|1-2\alpha|+9}}{2\beta|1-2\alpha|}, & \alpha\ne\frac{1}{2}, \\ \frac{1}{3}, & \alpha=\frac{1}{2}. \end{cases}$$

Using the following best possible inclusion relationship [18]

$$\mathscr{C}(\alpha) \subset \mathscr{S}^{\star}\left(\varkappa(\alpha)\right),$$

where

$$\varkappa(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2(1-\alpha)}(1-2^{2\alpha-1})}, & \alpha \neq \frac{1}{2}, \\ \frac{1}{2\ln 2}, & \alpha = \frac{1}{2}, \end{cases} \tag{3.16}$$

and taking $\beta = 1$ in Corollary 3.4, we obtain

Corollary 3.5. *Let the function f be in the class $\mathscr{A}$. If the function $g \in \mathscr{C}(\alpha)$ and $f \ll g$ in $\mathbb{U}$, then*

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r(\varkappa(\alpha)),$$

where

$$r(\varkappa(\alpha)) = \begin{cases} \frac{3+|1-2\varkappa(\alpha)|-\sqrt{|1-2\varkappa(\alpha)|^2+2|1-2\varkappa(\alpha)|+9}}{2|1-2\varkappa(\alpha)|}, & \varkappa(\alpha) \neq \frac{1}{2}, \\ \frac{1}{3}, & \varkappa(\alpha) = \frac{1}{2} \end{cases}$$

and $\varkappa(\alpha)$ is given by (3.16).

Remark. (i) For the choice $\beta = 1$ in Corollary 3.4, we get the result due to Altintaş et al. [1, Theorem 1], which in turn yields the corresponding work of MacGregor [19, Theorem 1B] for $\alpha = 0$.

(ii) Corollary 3.5 improves the corresponding result obtained by Altintaş et al. [1, Theorem 2].

(iii) For the case $\alpha = 0$, Corollary 3.5 yields a result due to MacGregor [19, Theorem 1C].

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