doi: 10.2478/v10062-011-0024-3

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXV, NO. 2, 2011 SECTIO A 191–202

MAGDALENA SOBCZAK-KNEĆ, VIKTOR V. STARKOV
and JAN SZYNAL

Old and new order of linear invariant family of harmonic mappings and the bound for Jacobian

Dedicated to the memory of Professor Jan G. Krzyż

Abstract. The relation between the Jacobian and the orders of a linear invariant family of locally univalent harmonic mapping in the plane is studied. The new order (called the strong order) of a linear invariant family is defined and the relations between order and strong order are established.

1. A harmonic mapping f in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ has a representation:

$$f(z) = h(z) + \overline{g(z)} \tag{1.1}$$

where h and g are holomorphic functions in $\mathbb{D}$.

We assume that f is locally univalent and sense-preserving in $\mathbb{D}$, which is equivalent to $J_f(z) > 0$, $z \in \mathbb{D}$, where $J_f(z)$ denotes the Jacobian of f:

$$J_f(z) = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 . \tag{1.2}$$

For the properties of harmonic mappings we can refer to surveys [1] and [2]. The notion of an affine and linear invariant family of univalent harmonic functions was proposed by Sheil-Small [6], and extended to local univalent mappings and then used efficiently by Schaubroeck in [5].

2000 *Mathematics Subject Classification.* Primary 30C55, Secondary 31A05.

Key words and phrases. Planar harmonic mappings, order of a linear invariant family, Jacobian.

For any holomorphic automorphism φ of $\mathbb{D}$ ($\varphi \in \mathrm{Aut}(\mathbb{D})$) we denote

$$T_\varphi\left(f(z)\right) = \frac{f(\varphi(z)) - f(\varphi(0))}{\varphi'(0)h'(\varphi(0))}, \tag{1.3}$$

$$A_\varepsilon(f(z)) = \frac{f(z) + \varepsilon\overline{f(z)}}{1 + \varepsilon g'(0)}, \quad |\varepsilon| < 1, \quad \varepsilon \in \mathbb{C}. \tag{1.4}$$

The transformations (1.3) and (1.4) are called the Koebe transform and the affine transform of a locally univalent harmonic function $f = h + \overline{g}$.

Put

$$S_{\varphi,\varepsilon}(f(z)) = A_\varepsilon \circ T_\varphi(f(z)) \tag{1.5}$$

In what follows L denotes a family of locally univalent and sense-preserving harmonic functions $f = h + \overline{g}$ in $\mathbb{D}$ which have the expansion:

$$f(z) = \sum_{n=2}^{\infty} a_n(f)z^n + z + \sum_{n=1}^{\infty} a_{-n}(f)\overline{z}^n, \qquad z \in \mathbb{D}. \tag{1.6}$$

A family L is called an affine and linear invariant $ALIF$ if for any $f \in L$ the function $T_\varphi(f)$ and $A_\varepsilon(f)$ belong to L for all $\varphi \in \mathrm{Aut}(\mathbb{D})$ and all $|\varepsilon| < 1$.

A family L is called LIF (linear invariant family) if for any $f \in L$ and all $\varphi \in \mathrm{Aut}(\mathbb{D})$ the function $T_\varphi(f) \in L$.

The order of the family L is defined as $\operatorname{ord} L = \sup\{|a_2(f)| : f \in L\}$ (see [5] and [6]).

Example. The best known $ALIF$ family is the class S_H of univalent harmonic mappings in $\mathbb{D}$ preserving orientation, as well as the subclasses K_H of convex and C_H of close-to-convex mapping [2].

A simple example of a family which is LIF but not $ALIF$ is the family of locally univalent holomorphic functions in $\mathbb{D}$.

The properties of the transformation (1.3)–(1.5) have been used in [5] to obtain some bounds for the Jacobian $J_f(z)$ in terms of the order of a linear invariant family.

In this paper we give an improvement of one result from [5] (Theorem 2.1) and establish the relations between $\operatorname{ord} L$ and the new order called the strong order $\overline{\operatorname{ord}}\, L$ defined below. Introduction of the new order $\overline{\operatorname{ord}}\, L$, allow us to prove Theorem 3.1 for arbitrary family L which is an extension of Theorem 2.1, while L is $ALIF$ family.

These relations depend on the upper bound for the Jacobian $J_f(z)$ in the terms of $\operatorname{ord} L$ and $\overline{\operatorname{ord}}\, L$.

We end this introduction with two definitions and one lemma.

Definition 1.1. The affine hull of the family L is defined as the set of functions

$$A(L) = \{A_\varepsilon(f) : f \in L, |\varepsilon| < 1\}.$$

Definition 1.2. The linear-affine hull of the family L is defined as the set of functions

$$\mathscr{L}A(L) = \{S_{\varphi,\varepsilon}(f) : f \in L,\ \varphi \in \mathrm{Aut}(\mathbb{D}),\ |\varepsilon| < 1\}.$$

Lemma 1.1. *If L is a linear invariant family of harmonic functions (LIF), then $A(L)$ is affine and linearly invariant $(ALIF)$.*

Proof. By the definition of $A(L)$ it is enough to prove that $A(L)$ is a (LIF). Every member of $A(L)$ has the form

$$f_\varepsilon(z) = \frac{f + \varepsilon\overline{f}}{1 + \varepsilon g'(0)} = h_\varepsilon + \overline{g_\varepsilon}, \quad \varepsilon \in \mathbb{C}, \quad |\varepsilon| < 1,$$

where $f(z) = h(z) + \overline{g(z)} \in L$. The functions h, g, h_ε and g_ε are holomorphic functions in $\mathbb{D}$ and

$$h_\varepsilon(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon g'(0)}, \quad g_\varepsilon(z) = \frac{g(z) + \overline{\varepsilon} h(z)}{1 + \overline{\varepsilon g'(0)}}.$$

We have to prove that $T_\varphi(f_\varepsilon) \in A(L)$ for any

$$\varphi(z) = e^{i\theta}\frac{z + a}{1 + \overline{a}z}, \quad a \in \mathbb{D}, \quad \theta \in \mathbb{R}.$$

Let us fix φ. Denote $F_\varepsilon = T_\varphi(f_\varepsilon) = H_\varepsilon + \overline{G_\varepsilon}$, where

$$H'_\varepsilon(z) = \frac{h'(\varphi(z)) + \varepsilon g'(\varphi(z))}{(h'(\varphi(0)) + \varepsilon g'(\varphi(0)))\,(1 + \overline{a}z)^2},$$

$$G'_\varepsilon(z) = \frac{g'(\varphi(z)) + \overline{\varepsilon} h'(\varphi(z))}{\overline{(h'(\varphi(0)) + \varepsilon g'(\varphi(0)))}(1 + \overline{a}z)^2} \cdot e^{2i\theta}.$$

Analogously, denote $T_\phi(f) = F = H + \overline{G} \in L$, where

$$H'(z) = \frac{h'(\varphi(z))}{h'(\varphi(0))(1 + \overline{a}z)^2}, \quad G'(z) = \frac{g'(\varphi(z))}{\overline{h'(\varphi(0))}(1 + \overline{a}z)^2} \cdot e^{2i\theta}$$

and

$$\varepsilon_1 = \varepsilon\frac{h'(\varphi(0))}{\overline{h'(\varphi(0))}} \cdot e^{2i\theta}.$$

We can write that $A_\varepsilon(F) \in A(L)$ because writing $A_\varepsilon(F) = \Phi_\varepsilon(z) = \widehat{H}_\varepsilon + \overline{\widehat{G}_\varepsilon}$, where

$$\widehat{H}'_\varepsilon(z) = \frac{h'(\varphi(z)) + \varepsilon_1 g'(\varphi(z))}{(h'(\varphi(0)) + \varepsilon_1 g'(\varphi(0)))\,(1 + \overline{a}z)^2} = H'_{\varepsilon_1}(z)$$

and

$$\widehat{G}'_\varepsilon(z) = \frac{g'(\varphi(z)) + \overline{\varepsilon_1} h'(\varphi(z))}{\overline{(h'(\varphi(0)) + \varepsilon_1 g'(\varphi(0)))}(1 + \overline{a}z)^2} \cdot e^{2i\theta} = G'_{\varepsilon_1}(z),$$

we have $\Phi_\varepsilon(z) = F_{\varepsilon_1} \in A(L)$, for any ε_1, $|\varepsilon_1| < 1$, due to the fact that ε was arbitrary and $|\varepsilon| < 1$. $\square$

Remark 1.1. As we see from the proof, the operators T_φ and A_ε do not commute, i.e. $A_\varepsilon \circ T_\varphi \neq T_\varphi \circ A_\varepsilon$. However, we have

$$\{A_\varepsilon \circ T_\varphi(f) : f \in L,\ |\varepsilon| < 1,\ \varphi \in \mathrm{Aut}(\mathbb{D})\} \\ = \{T_\varphi \circ A_\varepsilon(f) : f \in L,\ |\varepsilon| < 1,\ \varphi \in \mathrm{Aut}(\mathbb{D})\}.$$

2. We start with a slight improvement of Theorem 3.3 from [5].

Theorem 2.1. *If L is ALIF, $\operatorname{ord} L = \alpha$, $\alpha > 0$ and $f \in L$, then*

$$\begin{aligned}\left(1 - |a_{-1}(f)|^2\right)\frac{(1-r)^{2\alpha-2}}{(1+r)^{2\alpha+2}} &\le J_f(z) \\ &\le \frac{(1+r)^{2\alpha-2}}{(1-r)^{2\alpha+2}}\left(1 - |a_{-1}(f)|^2\right), \quad |z| = r < 1.\end{aligned} \tag{2.1}$$

The bounds in (2.1) *are sharp and the sign of equality holds for the function*

$$f(z) = k_\alpha(z) + a_{-1}\overline{k_\alpha(z)}, \tag{2.2}$$

where

$$k_\alpha(z) = \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^\alpha - 1\right].$$

Observe that $f(z)$ is univalent for $\alpha \in (0, 2]$, which follows from univalence of $k_\alpha(z)$ for these α and the invariance of univalent harmonic functions w.r.t. operator A_ε.

Proof. (Theorem 2.1) The proof is exactly the same as in [5], only the value $J_f(0) = 1 - |a_{-1}(f)|^2$ has to be taken into account. Namely, using the inequality from [5]:

$$\begin{aligned}\frac{d}{dr}\log\left(\frac{1-r}{1+r}\right)^{2\alpha} &\le \frac{\partial}{\partial r}\left[\log\left(J_f(re^{i\theta}) \cdot (1-r^2)^2\right)\right] \\ &\le \frac{d}{dr}\log\left(\frac{1+r}{1-r}\right)^{2\alpha}, \quad z = re^{i\theta},\end{aligned}$$

after integration along the segment $[0, r]$, $0 < r < 1$, we obtain (2.1).

For a linear invariant family L of holomorphic functions the inverse theorem holds (see [4]), i.e. inequality (2.1) implies that $\operatorname{ord} L \le \alpha$. $\square$

The next theorem is in some sense inverse to Theorem 2.1. We do not assume even the linear invariance of the family L.

Theorem 2.2. *Let $f \in L$ and assume that the upper bound in* (2.1) *holds for some $\alpha > 0$. Then there exists ε, $|\varepsilon| < 1$ such that*

$$\left|a_2(f) - \frac{\varepsilon}{2}\right| \le \alpha. \tag{2.3}$$

The inequality (2.3) *is sharp, which means that in the right side of the inequality $|\varepsilon| < 1$ we can not write any constant smaller than* 1.

Proof. We will apply the same ideas from [3]. By the assumption, $f \in L$ satisfies (2.1) which implies that for $z = re^{i\theta}$,

$$\log J_f(z) - \log J_f(0) \leq (2\alpha - 2)\log(1+r) - (2\alpha+2)\log(1-r).$$

For $r = 0$ the above inequality after differentiation gives

$$\frac{1}{J_f(0)} \cdot \frac{\partial}{\partial r} J_f(re^{i\theta})\Big|_{r=0} \leq 4\alpha. \tag{2.4}$$

But

$$\frac{\partial}{\partial r} J_f(z) = \left[h''(re^{i\theta})\overline{h'(re^{i\theta})}e^{i\theta} - g''(re^{i\theta})\overline{g'(re^{i\theta})}e^{i\theta}\right] + \overline{\left[h''(re^{i\theta})\overline{h'(re^{i\theta})}e^{i\theta} - g''(re^{i\theta})\overline{g'(re^{i\theta})}e^{i\theta}\right]}.$$

Therefore, by (2.4) for every real θ we have

$$\frac{\partial}{\partial r}\left(J_f(re^{i\theta})\right)\Big|_{r=0} = 2\,\mathrm{Re}\left\{e^{i\theta}\left(h''(0)\overline{h'(0)} - g''(0)\overline{g'(0)}\right)\right\} \leq 4\alpha J_f(0).$$

Thus

$$\left|h''(0) - \overline{g'(0)}g''(0)\right| \leq 2\alpha J_f(0),$$

due to the fact that $h'(0) = 1$. The above inequality is equivalent to

$$\frac{\left|a_2(f) - a_{-1}(f)\overline{a_{-2}(f)}\right|}{1 - |a_{-1}(f)|^2} \leq \alpha. \tag{2.5}$$

Let us put

$$\omega(z) = \frac{g'(z)}{h'(z)} = \overline{a_{-1}}(f) + 2(\overline{a_{-2}}(f) - a_2(f)\,\overline{a_{-1}}(f))z + \ldots$$

and

$$\omega_0(z) = \frac{\omega(z) - \overline{a_{-1}}(f)}{1 - a_{-1}(f)\omega(z)} = \frac{2(\overline{a_{-2}}(f) - a_2(f)\,\overline{a_{-1}}(f))}{1 - |a_{-1}(f)|^2} z + \ldots\ .$$

Because $\omega_0(0) = 0$ and $|\omega_0(z)| < 1$ we have

$$2\left|\frac{\overline{a_{-2}}(f) - a_2(f)\overline{a_{-1}}(f)}{1 - |a_{-1}(f)|^2}\right| \leq 1. \tag{2.6}$$

On the other hand, (2.5) can be rewritten as

$$\left|\frac{a_2(f)(1 - |a_{-1}(f)|^2) + a_{-1}(f)(a_2(f)\overline{a_{-1}}(f) - \overline{a_{-2}}(f))}{1 - |a_{-1}(f)|^2}\right| \leq \alpha$$

or

$$\left|a_2(f) - \frac{\varepsilon}{2}\right| \leq \alpha \quad \text{where} \quad |\varepsilon| = \left|a_{-1}(f)2\frac{a_2(f)\overline{a_{-1}(f)} - \overline{a_{-2}(f)}}{1 - |a_{-1}(f)|^2}\right| < 1.$$

In order to prove the sharpness consider the function

$$f_0(z) = h_0(z) + \overline{g_0(z)},$$

where

$$h_0(z) = k_{\alpha+\frac{1}{2}}(z), \quad g_0'(z) = k'_{\alpha+\frac{1}{2}}(z)\frac{z+x}{1+xz}, \quad x \in (0,1).$$

We have $a_2(f_0) = \alpha + \frac{1}{2}$ and

$$\frac{J_{f_0}(z)}{J_{f_0}(0)} = \frac{\left|k'_{\alpha+\frac{1}{2}}(z)\right|^2 \left(1 - \left|\frac{z+x}{1+xz}\right|^2\right)}{\left|k'_{\alpha+\frac{1}{2}}(0)\right|^2 (1-x^2)} = \left|\frac{(1+z)^{2(\alpha-\frac{1}{2})}}{(1-z)^{2(\alpha+\frac{3}{2})}}\right| \frac{1-|z|^2}{|1+xz|^2}.$$

Moreover,

$$\begin{aligned}\min_{|z|=r}\left(|1-z|^2|1+xz|^2\right) &= \min_{\theta\in\mathbb{R}}\left[(1-2r\cos\theta+r^2)(1+2xr\cos\theta+x^2r^2)\right] \\ &= (1-r)^2(1+xr)^2,\end{aligned}$$

and the minimum is attained for $\theta = 0$. Therefore,

$$\max_{|z|=r}\frac{J_{f_0}(z)}{J_{f_0}(0)} = \frac{(1+r)^{2\alpha-1}}{(1-r)^{2\alpha+3}} \cdot \frac{(1-r)^2}{(1+xr)^2} < \frac{(1+r)^{2\alpha-2+2\delta(x)}}{(1-r)^{2\alpha+2+\delta(x)}},$$

which implies that inequality (2.3) for f_0 can be written in the form

$$\left|\left(\alpha+\frac{1}{2}\right) - \frac{\varepsilon}{2}\right| \le \alpha + \delta(x),$$

where $0 < \delta(x) \to 0$ if $x \to 1^-$, and

$$\varepsilon = -a_{-1}(f_0)2\frac{a_2(f_0)\overline{a_{-1}(f_0)} - \overline{a_{-2}(f_0)}}{1-|a_{-1}(f_0)|^2} = x,$$

because $2\overline{a_{-2}(f_0)} = 1 - x^2 + 2a_2(f_0)x$. This makes the result of Theorem 2.2 sharp. □

Remark 2.1. Theorem 2.2 is also valid for holomorphic functions $f(z)$. In this case we have to put in the proof $\varepsilon = 0$.

Remark 2.2. From the above proof we see that in the statement of Theorem 2.2 it is sufficient to assume that $f(z)$ satisfies only the right- (or left-) hand side of inequality (2.1).

Corollary 2.1. *If the family L is LIF and for any $f \in L$ inequality* (2.1) *holds, then* $\operatorname{ord} L \le \alpha + \frac{1}{2}$.

3. Now we introduce the definition of new order $\overline{\operatorname{ord}}\, L$ (we will call the strong order) of a linear invariant family (LIF) of harmonic mappings L. In terms of this new order one can formulate iff version of Theorem 2.1 without assuming family L to be affine.

Definition 3.1. Let L be LIF. The strong order $\overline{\operatorname{ord}}\, L$ of a family L of harmonic mappings f is defined by the formula

$$\overline{\operatorname{ord}}\, L = \sup_{f\in L} \frac{|a_2(f) - a_{-1}(f)\overline{a_{-2}}(f)|}{1-|a_{-1}(f)|^2}. \tag{3.1}$$

Remark 3.1. In the case when f is holomorphic in $\mathbb{D}$, definition (3.1) coincides with that introduced by Pommerenke in [4].

Definition 3.2. For any fixed $f \in L$ we define

$$\overline{\operatorname{ord}}\, f = \sup_{\varphi\in \operatorname{Aut}(\mathbb{D})} \frac{|a_2\left(T_\varphi(f)\right) - a_{-1}\left(T_\varphi(f)\right)\overline{a_{-2}}\left(T_\varphi(f)\right)|}{1-|a_{-1}\left(T_\varphi(f)\right)|^2}. \tag{3.2}$$

Of course, if L is LIF, $\overline{\operatorname{ord}}\, L = \alpha$, then $\overline{\operatorname{ord}}\, f \le \alpha$ for any $f \in L$.

Theorem 3.1. *If $f \in L$, then $\overline{\operatorname{ord}}\, f \le \alpha$ if and only if for every $F = F_\psi := T_\psi(f)$ and any $z \in \mathbb{D}$*

$$\frac{(1-r)^{2\alpha-2}}{(1+r)^{2\alpha+2}} \le \frac{J_F(z)}{J_F(0)} \le \frac{(1+r)^{2\alpha-2}}{(1-r)^{2\alpha+2}}, \quad |z| = r < 1. \tag{3.3}$$

Proof. Assume first that $\overline{\operatorname{ord}}\, f \le \alpha$, $\varphi(z) = \frac{z+a}{1+\overline{a}z}$, $z, a \in \mathbb{D}$ and $T_\psi(f) = F = H + \overline{G}$, $\psi \in \operatorname{Aut}(\mathbb{D})$. Consider $F_a(z) = T_\varphi(F) = H_a + \overline{G}_a$, where H, H_a, G and G_a are holomorphic functions in $\mathbb{D}$. By direct calculations we find

$$G_a'(z) = \frac{G'\left(\varphi(z)\right)}{\overline{H'(a)}\cdot(1+\overline{a}z)^2}, \qquad H_a'(z) = \frac{H'\left(\varphi(z)\right)}{H'(a)\cdot(1+\overline{a}z)^2},$$

and

$$G_a'(0) = \frac{G'(a)}{\overline{H'(a)}}, \qquad G_a''(0) = \frac{G''(a)(1-|a|^2) - 2\overline{a}G'(a)}{\overline{H'(a)}},$$

$$H_a''(0) = \frac{H''(a)(1-|a|^2)}{H'(a)} - 2\overline{a}.$$

One can easily verify that

$$J_{F_a}(z) = \frac{J_F\left(\varphi(z)\right)}{|H'(a)|^2|1+\overline{a}z|^4}.$$

Moreover, for $z = re^{i\theta} \in \mathbb{D}$ we have

$$\left.\frac{\partial J_{F_a}(re^{i\theta})}{\partial r}\right|_{r=0} = 2\operatorname{Re}\left\{e^{i\theta}\left(H_a''(0) - G_a''(0)\overline{G_a'(0)}\right)\right\}, \tag{3.4}$$

and by the definition of the order

$$\begin{aligned}\left.\frac{\partial J_{F_a}(re^{i\theta})}{\partial r}\right|_{r=0} &= 2\operatorname{Re}\left\{e^{i\theta}\frac{\partial}{\partial z}J_{F_a}(z)\right\}\Big|_{z=0} \\ &= 2\operatorname{Re}\left\{e^{i\theta}(2a_2(F_a) - 2\overline{a_{-2}}(F_a)a_{-1}(F_a)\right\} \\ &\le 4\alpha(1-|a_{-1}(F_a)|^2).\end{aligned} \tag{3.5}$$

Therefore, from (3.4) we obtain

$$\begin{aligned}\left.\frac{\partial J_{F_a}(re^{i\theta})}{\partial r}\right|_{r=0} &= 2\operatorname{Re}\left\{e^{i\theta}\left[\frac{H''(a)(1-|a|^2)}{H'(a)} - 2\overline{a}\right.\right. \\ &\qquad \left.\left. - \frac{\overline{G'(a)}}{|H'(a)|^2}\Big(G''(a)(1-|a|^2) - 2\overline{a}G'(a)\Big)\right]\right\} \\ &= 2\operatorname{Re}\left\{e^{i\theta}\left[\frac{(1-|a|^2)}{|H'(a)|^2}\cdot\frac{\partial J_F(a)}{\partial z} - 2\overline{a}\left(1-\left|\frac{G'(a)}{H'(a)}\right|^2\right)\right]\right\}.\end{aligned}$$

Choosing first $\theta = a$, $a \neq 0$, we obtain ($a = \varrho e^{i\theta}$)

$$\left.\frac{\partial J_{F_a}(re^{i\theta})}{\partial r}\right|_{r=0} = \frac{1}{|H'(a)|^2}\left[(1-|a|^2)\frac{\partial J_F(\varrho e^{i\theta})}{\partial\varrho} - 4|a|J_F(\varrho e^{i\theta})\right].$$

Applying (3.5), we get

$$\frac{1}{J_F(\varrho e^{i\theta})}\frac{\partial J_F}{\partial\varrho}(\varrho e^{i\theta}) \le \frac{4(\alpha+\varrho)}{1-\varrho^2}.$$

Choosing now $\theta = \pi + a$, we obtain

$$\frac{1}{J_F(\varrho e^{i\theta})}\cdot\frac{\partial J_F(\varrho e^{i\theta})}{\partial\varrho} \ge 4\frac{\varrho-\alpha}{1-\varrho^2}.$$

Writing together the above inequalities, we get

$$\begin{aligned}2\frac{\partial}{\partial\varrho}\left(\log(1-\varrho)^{\alpha-1} - \log(1+\varrho)^{\alpha+1}\right) &\le \frac{\partial(\log J_F(\varrho e^{i\theta}))}{\partial\varrho} \\ &\le 2\frac{\partial}{\partial\varrho}\left(\log(1+\varrho)^{\alpha-1} - \log(1-\varrho)^{\alpha+1}\right).\end{aligned}$$

Integration over the interval $[0,r]$ implies (3.3).

Assume now that the right-hand side of (3.3) holds for some $\alpha > 0$ and $r \in (0,1)$. We have to prove now that $\overline{\operatorname{ord}} f \le \alpha$.

Because for the function $F = T_\varphi(f)$ the inequality (3.3) holds, therefore we have

$$\log J_F(z) - \log J_F(0) \le (2\alpha-2)\log(1+r) - (2\alpha+2)\log(1-r).$$

This implies, as in the proof of Theorem 2.2, the validity of equality (2.5) for the function $F = T_\varphi(f)$, and we have $\overline{\operatorname{ord}} f \le \alpha$. $\square$

From the proof of the above theorem we obtain the following corollary.

Corollary 3.1. *If L is LIF, then $\overline{\operatorname{ord}}\, L \le \alpha$ if and only if the right-hand side of* (3.3) *holds for any $f \in L$ and for every $r \in (0, \delta)$, $0 < \delta < 1$.*

If L is LIF, then by Theorem 3.1 we can give an equivalent definition of the $\overline{\operatorname{ord}}\, L$. Namely,

Corollary 3.2.

$$\overline{\operatorname{ord}}\, L = \inf\left\{\alpha : \frac{J_f(z)}{J_f(0)} \le \frac{(1+|z|)^{2\alpha-2}}{(1-|z|)^{2\alpha+2}} \quad \text{for any } f \in L \text{ and any } z \in \mathbb{D}\right\}. \tag{3.6}$$

From Theorem 2.1 and Corollary 2.1 we derive the next corollaries.

Corollary 3.3. *If L is LIF, then*

$$\operatorname{ord} L \le \overline{\operatorname{ord}}\, L + \tfrac{1}{2}.$$

Corollary 3.4. *If L is LIF, then*

$$\overline{\operatorname{ord}} L \le \operatorname{ord} A(L), \quad \operatorname{ord} A(L) \ge \operatorname{ord} L. \tag{3.7}$$

If L is $ALIF$, then

$$\operatorname{ord} L - \tfrac{1}{2} \le \overline{\operatorname{ord}}\, L \le \operatorname{ord} L. \tag{3.8}$$

In particular, because the class S_H of harmonic univalent functions in $\mathbb{D}$ is an $ALIF$, we have

$$\operatorname{ord} S_H - \tfrac{1}{2} \le \overline{\operatorname{ord}}\, S_H \le \operatorname{ord} S_H.$$

Indeed, since $A(L)$ is a LIF, then by Theorem 2.1 for any $f \in A(L)$ inequality (3.3) holds with $\alpha = \operatorname{ord} A(L)$ and we have $\overline{\operatorname{ord}}\, L \le \operatorname{ord} A(L)$. The second inequality $\operatorname{ord} A(L) \ge \operatorname{ord} L$ follows by the relation $L \subset A(L)$.

From (3.7) and the fact that $\mathscr{L}A(L) = L$ if L is an $ALIF$, we have (3.8).

Remark 3.2. The equality $\operatorname{ord} L = \overline{\operatorname{ord}}\, L$ is possible. Take for example $L = \mathscr{L}A(k_\alpha)$, where k_α is given by (2.2). However, the inequality $\operatorname{ord} L \ne \overline{\operatorname{ord}}\, L$ can hold as well, as shows the following.

Example. Let $k_\alpha(z)$ be the generalized Koebe function given by (2.2).

Put $f_\alpha(z) = k_\alpha(z) + \overline{g(z)}$, where $g'(z) = zk'_\alpha(z)$. Consider the family $L = \mathscr{L}(f_\alpha)$, the linear– invariant hull of the function f_α, i.e.

$$\mathscr{L}(f_\alpha) = \{T_\varphi(f_\alpha) : \varphi \in \operatorname{Aut} \mathbb{D}\}.$$

If $F \in \mathscr{L}(f_\alpha)$, then there exists $\varphi \in \operatorname{Aut}(\mathbb{D})$ such that

$$F(z) = \frac{k_\alpha(\varphi(z)) - k_\alpha(\varphi(0))}{k'_\alpha(\varphi(0))\varphi'(0)} + \frac{\overline{g(\varphi(z))} - \overline{g(\varphi(0))}}{k'_\alpha(\varphi(0))\varphi'(0)}.$$

Denote $\operatorname{ord} F = \operatorname{ord} \mathscr{L}(F)$. Therefore, $\operatorname{ord} F = \operatorname{ord} k_\alpha = \alpha$ (see [4]) and $\operatorname{ord} L = \alpha$.

Taking $\varphi(z) = \frac{z+a}{1+\overline{a}z}$, $(a \in \mathbb{D})$ we find

$$\begin{aligned} \frac{J_F(z)}{J_F(0)} &= \left| \frac{\left(1+\frac{z+a}{1+\overline{a}z}\right)^{\alpha-1} \cdot (1-a)^{\alpha+1}}{\left(1-\frac{z+a}{1+\overline{a}z}\right)^{\alpha+1} \cdot (1+a)^{\alpha-1} \cdot (1+\overline{a}z)^2} \right|^2 \frac{\left(1-\left|\frac{z+a}{1+\overline{a}z}\right|^2\right)}{1-|a|^2} \\ &= \left| \frac{\left(1+z\frac{1+\overline{a}}{1+a}\right)^{\alpha-1}}{\left(1-z\frac{1-\overline{a}}{1-a}\right)^{\alpha+1}} \right|^2 \cdot \frac{1-|z|^2}{|1+\overline{a}z|^2}. \end{aligned} \tag{3.9}$$

For $z = r > 0$ and $a \in (-1, 0)$, when $a \to -1$ the right-hand side of the latter expression tends to $\frac{(1+r)^{2\alpha-1}}{(1-r)^{2\alpha+3}}$ and therefore, $\overline{\operatorname{ord}}\, L \geq \alpha + \frac{1}{2}$ (in fact $\overline{\operatorname{ord}}\, L = \alpha + \frac{1}{2}$). So $\overline{\operatorname{ord}}\, L \neq \operatorname{ord} L$.

Theorem 3.2. *Assume that $f_1, f_2 \in L$ and L is LIF. If*

$$\frac{J_{f_1}(z)}{J_{f_1}(0)} = \frac{J_{f_2}(z)}{J_{f_2}(0)}$$

for every $z \in \mathbb{D}$, then $\overline{\operatorname{ord}}\, f_1 = \overline{\operatorname{ord}}\, f_2$.

Proof. We conclude the proof of Theorem 3.2 by Corollary 3.2 because

$$\begin{aligned} &\inf\{\alpha : (3.3) \text{ holds for any } F \in \mathscr{L}(f_1)\} \\ &\qquad = \inf\{\alpha : (3.3) \text{ holds for any } F \in \mathscr{L}(f_2)\}, \end{aligned}$$

where $\mathscr{L}(f)$ is defined as above in the Example. □

Definition 3.3. Put

$$U_\alpha^H = \bigcup \{L : L \text{ is } LIF \text{ and } \overline{\operatorname{ord}}\, L \leq \alpha\}$$

and call it the universal LIF of strong order α.

Remark 3.3. From the definition it is obvious that $f \in U_\alpha^H$ iff $\overline{\operatorname{ord}}\, f \leq \alpha$.

Theorem 3.3. *The family U_α^H is ALIF.*

Proof. We have to prove that for any $f \in U_\alpha^H$ and $|\varepsilon| < 1$, the function

$$A_\varepsilon(f) = f_\varepsilon(z) = \frac{f(z) + \varepsilon\overline{f(z)}}{1 + \varepsilon\overline{a_{-1}}(f)} \in U_\alpha^H,$$

i.e. $\overline{\operatorname{ord}}\, f_\varepsilon \leq \alpha$. Putting

$$\varepsilon = \varrho e^{i\beta}, \quad \varrho \in [0, 1), \quad \beta \in \mathbb{R},$$

we can find that

$$\frac{J_{f_\varepsilon}(z)}{J_{f_\varepsilon}(a)} = \frac{J_f(z)}{J_f(a)}, \quad z, a \in \mathbb{D}. \tag{3.10}$$

Indeed, we have for any harmonic function $f = u + iv$, where u and v are real functions: $J_f = u_x v_y - u_y v_x$. Therefore, for $f_\varepsilon = A_\varepsilon(f)$ we obtain

$$\begin{aligned} &J_{f_\varepsilon}(z) = \\ &= \frac{1}{|1+\varepsilon\overline{a_{-1}}(f)|^2}\begin{vmatrix} u_x(1+\varrho\cos\beta)+v_x\varrho\sin\beta & u_y(1+\varrho\cos\beta)+v_y\varrho\sin\beta \\ v_x(1-\varrho\cos\beta)+u_x\varrho\sin\beta & v_y(1-\varrho\cos\beta)+u_y\varrho\sin\beta \end{vmatrix} \\ &= \frac{(1-\varrho^2\cos^2\beta)J_f(z)}{|1+\varepsilon\overline{a_{-1}}(f)|^2}, \end{aligned}$$

which implies (3.10). By Corollary 3.2 putting $|z| = r$, we have

$$\overline{\operatorname{ord}}\, f_\varepsilon = \inf\left\{\alpha : \frac{J_{F_\varepsilon}(z)}{J_{F_\varepsilon}(0)} \le \frac{(1+r)^{2\alpha-2}}{(1-r)^{2\alpha+2}} \text{ for any } F_\varepsilon = T_\varphi(f_\varepsilon),\ \varphi \in \operatorname{Aut}(\mathbb{D})\right\}.$$

But from (3.10)

$$\frac{J_{F_\varepsilon}(z)}{J_{F_\varepsilon}(0)} = \frac{J_{f_\varepsilon}(\varphi(z))}{J_{f_\varepsilon}(\varphi(0))|1-\overline{\varphi^{-1}(0)}z|^4} = \frac{J_F(z)}{J_F(0)}, \quad F = T_\varphi(f).$$

Therefore,

$$\begin{aligned} \overline{\operatorname{ord}}\, f_\varepsilon &= \inf\left\{\alpha : \frac{J_F(z)}{J_F(0)} \le \frac{(1+r)^{2\alpha-2}}{(1-r)^{2\alpha+2}} \text{ for any } F = T_\varphi(f),\ \varphi \in \operatorname{Aut}(\mathbb{D})\right\} \\ &= \overline{\operatorname{ord}}\, f \le \alpha, \end{aligned}$$

because $f \in U_\alpha^H$. $\square$

Corollary 3.5. *If L is LIF, then*

$$\overline{\operatorname{ord}}\, L \ge 1.$$

Proof. Assume on the contrary that $\overline{\operatorname{ord}}\, L < 1$. Then from the left-hand side of (3.3) it follows that

$$\frac{J_F(z)}{J_F(0)} \to +\infty, \qquad \text{as} \qquad |z| = r \to 1^-.$$

This implies that the numerator of the above expression, which is $|h'(z)|^2 - |g'(z)|^2 \to +\infty$, and therefore, $|h'(z)| \to +\infty$. This is in contradiction with the minimum principle because we would have

$$\min_{|z|=r<1} |h'(z)| \le 1,$$

due to the fact that $h'(0) = 1$. $\square$

Acknowledgements. The authors would like to express their great thanks to the Referee, whose detailed and careful review helped to improve the presentation of the results.

The second author has been partially supported by the RFBR Grant No. 11-01-00952a.

References

[1] Bshouty, D., Hengartner, W., *Univalent harmonic mappings in the plane*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **48** (1994), 13–42.
[2] Duren, P., *Harmonic Mappings in the Plane*, Cambridge University Press, Cambridge, 2004.
[3] Godula, J., Liczberski, P. and Starkov, V. V., *Order of linearly invariant mappings in* C^n, Complex Variables Theory Appl. **42** (2000), 89–96.
[4] Pommerenke, Ch., *Linear-invariante Familien analytischer Funktionen I*, Math. Annalen **155** (1964), 108–154.
[5] Schaubroeck, L. E., *Subordination of planar harmonic functions*, Complex Variables Theory Appl. **41**, (2000), 163–178.
[6] Sheil-Small, T., *Constants for planar harmonic mappings*, J. London Math. Soc. **42** (1990), 237–248.

M. Sobczak-Kneć
Lublin University of Technology
ul. Nadbystrzycka 38D
20-618 Lublin
Poland
e-mail: m.sobczak-knec@pollub.pl

V. V. Starkov
Department of Mathematics
University of Petrozavodsk
ul. Lenina 33
185910 Petrozavodsk
Russia
e-mail: vstar@psu.karelia.ru

J. Szynal
Department of Mathematics
University of Economics and Innovations
ul. Mełgiewska 7-9
20–209 Lublin
Poland
e-mail: jan.szynal3@gmail.com

Received July 4, 2010