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On compactness and connectedness of the paratingent

Abstract. In this note we shall prove that for a continuous function $\varphi : \Delta \to \mathbb{R}^n$, where $\Delta \subset \mathbb{R}$, the paratingent of φ at $a \in \Delta$ is a non-empty and compact set in $\mathbb{R}^n$ if and only if φ satisfies Lipschitz condition in a neighbourhood of a. Moreover, in this case the paratingent is a connected set.

1. Notations and definitions. Let $\mathbb{R}$ denote a real line, $\Delta \subset \mathbb{R}$ an interval and $\mathbb{R}^n$ the Euclidean n-dimensional space with usual norm

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2},$$

where $x = (x_1, x_2, \ldots, x_n)$. The symbol

$$\langle\langle A \rangle\rangle = \sup \left\{ \|x\| : x \in A \right\}$$

is defined for any $A \subset \mathbb{R}^n$, $A \neq \emptyset$. The differential quotient $\frac{\varphi(t)-\varphi(s)}{t-s}$, where $\varphi : \Delta \to \mathbb{R}^n$ is a continuous function, $t, s \in \Delta$ and $t < s$, is denoted by $\mathcal{D}q\,(t,s)$. Let $u^\alpha = (1-\alpha)t + \alpha\tau$ and $v^\alpha = (1-\alpha)s + \alpha\sigma$ for $t, s, \tau, \sigma \in \Delta$, $t < s$, $\tau < \sigma$ and $\alpha \in [0,1]$. Evidently, $u^\alpha < v^\alpha$.

The set of all points $x \in \mathbb{R}^n$ for which there exist two sequences $\{t_k\}, \{s_k\} \subset \Delta$ such that $t_k < s_k$, both sequences converge to a and

$$x = \lim_{k\to\infty} \mathcal{D}q\,(t_k, s_k),$$

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is called the paratingent of φ at a and is denoted by $(\mathcal{P}\varphi)(a)$.

We shall say that a function $\varphi : \Delta \to \mathbb{R}^n$ satisfies Lipschitz condition in a neighbourhood of a point $a \in \Delta$, if

$$\exists_{L>0}\, \exists_{\delta>0}\, \forall_{t,s\in\Delta,\ |t-a|<\delta,\ |s-a|<\delta}\, \|\varphi(t) - \varphi(s)\| \le L\, |t-s|.$$

The distance of a point x from set A is denoted by

$$\delta(x, A) = \inf \big\{ \|x - y\| : y \in A \big\}.$$

2. Theorems.

Theorem 2.1. *The paratingent $(\mathcal{P}\varphi)(a)$ is a closed set in $\mathbb{R}^n$.*

Proof. Let $x^m \in (\mathcal{P}\varphi)(a)$, $m = 1, 2, \ldots$, and $\lim_{m\to\infty} x^m = x$. So we have

$$x^m = \lim_{k\to\infty} \mathcal{D}q\,(t_k^m, s_k^m)\,,$$

where $t_k^m, s_k^m \in \Delta$, $t_k^m < s_k^m$, $\lim_{k\to\infty} t_k^m = \lim_{k\to\infty} s_k^m = a$ and $m = 1, 2, \ldots$.

Then there exists k_m for any m such that $|t_{k_m}^m - a| < \frac{1}{m}$, $|s_{k_m}^m - a| < \frac{1}{m}$, and $\big\|\mathcal{D}q\,\big(t_{k_m}^m, s_{k_m}^m\big) - x^m\big\| < \frac{1}{m}$. Hence

$$x = \lim_{m\to\infty} \mathcal{D}q\,\big(t_{k_m}^m, s_{k_m}^m\big)\,,$$

where $\lim_{k\to\infty} t_{k_m}^m = \lim_{k\to\infty} s_{k_m}^m = a$. Thus $x \in (\mathcal{P}\varphi)(a)$, so $(\mathcal{P}\varphi)(a)$ is closed. $\square$

Theorem 2.2. *The paratingent $(\mathcal{P}\varphi)(a)$ is a non-empty and compact set if and only if the function φ satisfies Lipschitz condition in a neighbourhood of a.*

Proof. $(\Leftarrow)$

Let φ satisfy Lipschitz condition, hence there exist $L > 0$ and $\delta > 0$ such that $\|\mathcal{D}q\,(t, s)\,\| \le L$ for any $t, s \in \Delta$, $|t - a| < \delta$ and $|s - a| < \delta$. Hence the paratingent $(\mathcal{P}\varphi)(a)$ is bounded. Thus, by Theorem 2.1, $(\mathcal{P}\varphi)(a)$ is compact.

Let now $t_k, s_k \to a$ with $t_k < s_k$. The sequence $\big\{\mathcal{D}q\,(t_k, s_k)\,\big\}$ is bounded, so it contains a convergent subsequence, i.e. $\lim_{m\to\infty} \mathcal{D}q\,(t_{k_m}, s_{k_m}) = x \in (\mathcal{P}\varphi)(a)$, hence $(\mathcal{P}\varphi)(a)$ is non-empty.

$(\Rightarrow)$

Let $(\mathcal{P}\varphi)(a)$ be non-empty and compact, and assume that φ does not satisfy Lipschitz condition in any neighbourhood of a.

Firstly, let $x \in (\mathcal{P}\varphi)(a)$. There exists M such that $\langle\!\langle(\mathcal{P}\varphi)(a)\rangle\!\rangle \le M$. As x belongs to $(\mathcal{P}\varphi)(a)$, we have $x = \lim_{k\to\infty} \mathcal{D}q\,(t_k, s_k)$ for some sequences $\{t_k\}, \{s_k\} \subset \Delta$, $t_k < s_k$ and $t_k, s_k \to a$. Hence there exists k_0 such that $\|\mathcal{D}q\,(t_k, s_k)\,\| < 2M$ for $k \ge k_0$.

On the other hand, as φ does not satisfy Lipschitz condition, there exist sequences $\{\tau_k\}, \{\sigma_k\} \subset \Delta$, $\tau_k < \sigma_k$ and $|\tau_k - a|, |\sigma_k - a| < \frac{1}{k}$ such that $\|\mathcal{D}q\,(\tau_k, \sigma_k)\,\| > 4M$ for $k = 1, 2, \ldots$.

Let now $\varrho_k(\alpha) = \|\mathcal{D}q\,(u_k^\alpha, v_k^\alpha)\|$, where $\alpha \in [0,1]$ and u^α, v^α were defined in the first section. Function $\varrho_k : [0,1] \to \mathbb{R}$ is continuous and such that $\varrho_k(0) < 2M$ and $\varrho_k(1) > 4M$ for any $k = 1, 2, \ldots$. Thus there exists a sequence $\alpha_k \in [0,1]$ such that $\varrho_k(\alpha_k) = \left\|\mathcal{D}q\left(u_k^{\alpha_k}, v_k^{\alpha_k}\right)\right\| = 3M$.

Of course $u_k^{\alpha_k}, v_k^{\alpha_k} \to a$ as k tends to infinity. The sequence of quotients $\mathcal{D}q\left(u_k^{\alpha_k}, v_k^{\alpha_k}\right)$ is bounded, hence it contains a subsequence $\mathcal{D}q\left(u_{k_m}^{\alpha_{k_m}}, v_{k_m}^{\alpha_{k_m}}\right)$ convergent to a point $y \in (\mathcal{P}\varphi)(a)$. But we have $\|y\| = 3M$, which contradicts the assumption $\|y\| \le M$ as $(\mathcal{P}\varphi)(a)$ is bounded by the constant M. Therefore φ must satisfy Lipschitz condition in some neighbourhood of a. □

Theorem 2.3. *If $\varphi : \Delta \to \mathbb{R}^n$ satisfies Lipschitz condition in a neighbourhood of $a \in \Delta$, then the paratingent $(\mathcal{P}\varphi)(a)$ is a continuum, i.e. it is a non-empty compact and connected set.*

Proof. By Theorem 2.2 it is enough to show that $(\mathcal{P}\varphi)(a)$ is connected.

Assume "*a contrario*" that $(\mathcal{P}\varphi)(a)$ is not connected, i.e. $(\mathcal{P}\varphi)(a) = E_0 \cup E_1$, where sets $\emptyset \ne E_i, i = 0, 1$ are compact and $E_0 \cap E_1 = \emptyset$. Then $d = \inf\left\{\|x - y\| : x \in E_0, y \in E_1\right\} > 0$.

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function given by the formula $g(x) = \delta(x, E_0) - \delta(x, E_1)$. Function g is continuous. Moreover, if $x \in E_0$, then $g(x) \le -d$, and if $x \in E_1$, then $g(x) \ge d$. Hence $g(x) \ne 0$ for all $x \in (\mathcal{P}\varphi)(a)$.

Let us now fix $x^0 \in E_0$ and $x^1 \in E_1$. So we have $x^0 = \lim_{k\to\infty} \mathcal{D}q\,(t_k, s_k)$ and $x^1 = \lim_{k\to\infty} \mathcal{D}q\,(\tau_k, \sigma_k)$ for some sequences $\{t_k\}, \{s_k\}, \{\tau_k\}, \{\sigma_k\} \subset \Delta$, $t_k < s_k$, $\tau_k < \sigma_k$ and

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \tau_k = \lim_{k\to\infty} \sigma_k = a.$$

There exists k_0 such that $g\big(\mathcal{D}q\,(t_k, s_k)\big) < -\frac{d}{2}$ and $g\big(\mathcal{D}q\,(\tau_k, \sigma_k)\big) > \frac{d}{2}$ for $k \ge k_0$.

Let us now consider a family of functions $h_k : [0,1] \to \mathbb{R}$, for $k \ge k_0$, given by the formula $h_k(\alpha) = g\big(\mathcal{D}q\,(u_k^\alpha, v_k^\alpha)\big)$. We have $h_k(0) = g\big(\mathcal{D}q\,(t_k, s_k)\big) < -\frac{d}{2} < 0$ and $h_k(1) = g\big(\mathcal{D}q\,(\tau_k, \sigma_k)\big) > \frac{d}{2} > 0$. There exists a sequence $\alpha_k \in [0,1]$ such that $h_k(\alpha_k) = 0$ for $k \ge k_0$. The sequence $\mathcal{D}q\left(u_k^{\alpha_k}, v_k^{\alpha_k}\right)$ is bounded, so it contains a subsequence $\mathcal{D}q\left(u_{k_m}^{\alpha_{k_m}}, v_{k_m}^{\alpha_{k_m}}\right)$ convergent to point $y \in (\mathcal{P}\varphi)(a) = E_0 \cup E_1$. Hence $g(y) \ne 0$, which contradicts the fact that

$$\begin{aligned} g(y) &= g\Big(\lim_{m\to\infty} \mathcal{D}q\left(u_{k_m}^{\alpha_{k_m}}, v_{k_m}^{\alpha_{k_m}}\right)\Big) = \lim_{m\to\infty} g\big(\mathcal{D}q\left(u_{k_m}^{\alpha_{k_m}}, v_{k_m}^{\alpha_{k_m}}\right)\big) \\ &= \lim_{m\to\infty} h_{k_m}(\alpha_{k_m}) = 0. \end{aligned}$$

Therefore, the set $(\mathcal{P}\varphi)(a)$ is connected, which completes the proof. □

3. Remarks. The definition of paratingent used in this note is an analytic modification by A. Bielecki [2] of the original definition given by G. Bouligand [3]. The Bouligand definition has a geometrical character and it applies to every general set $E \subset \mathbb{R}^n$. Let us recall this definition (cf. [4, Def. VII.1.1]):

Definition. In the Euclidean space $\mathbb{R}^n$ the direction of a half-line (or in other words a ray $xy^{\rightarrow}$) with origin at a point x and passing through a point y is identified in the well-known way with a point of the unit sphere in $\mathbb{R}^n$. This identification gives us the topological structure in the set of all directions (i.e. rays).

Paratingent of the set $E \subset \mathbb{R}^n$ at point $x \in E$ is the set $(\mathcal{P}_E)(x)$ of all limits of the directions of sequences of half-lines $y_k z_k^{\rightarrow}$, where $y_k, z_k \in E$ and $y_k, z_k \to x$.

If a point x is an accumulation point of the set E, then the paratingent $(\mathcal{P}_E)(x)$ is always compact and non-empty set (cf. [4, Proposition VII.1.2]). So let $\varphi : \Delta \to \mathbb{R}^n$ be a given continuous function. Then the paratingent in the Bouligand sense of the function φ at point $a \in \Delta$ is the set $(\mathcal{P}_{\mathrm{Gr}\varphi})((a, \varphi(a)))$, where $\mathrm{Gr}\varphi = \big\{(t, \varphi(t)) : t \in \Delta\big\} \subset \mathbb{R}^{1+n}$ is the graph of the function φ. Of course the set $(\mathcal{P}_{\mathrm{Gr}\varphi})((a, \varphi(a)))$ is always non-empty and compact in $\mathbb{R}^{n+1}$.

Instead, the paratingent presented in this note (i.e. in Bielecki sense) of a function φ at a point a, i.e. the set $(\mathcal{P}\varphi)(a) \subset \mathbb{R}^n$, can be empty, bounded or unbounded.

Examples: Let $\varphi : \mathbb{R} \to \mathbb{R}$.
(1) $\varphi(t) = t^{1/3}$, then $(\mathcal{P}\varphi)(0) = \emptyset$, but

$$(\mathcal{P}_{\mathrm{Gr}\varphi})(0, \varphi(0)) = \big\{(0, -1), (0, 1)\big\} \subset \mathbb{R}^2;$$

(2) $\varphi(t) = |t|$, then $(\mathcal{P}\varphi)(0) = [-1, 1] \subset \mathbb{R}$, but

$$(\mathcal{P}_{\mathrm{Gr}\varphi})(0, \varphi(0)) = \left\{(\cos t, \sin t) : t \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3}{4}\pi, \frac{5}{4}\pi\right]\right\} \subset \mathbb{R}^2;$$

(3) $\varphi(t) = \sqrt{|t|}$, then $(\mathcal{P}\varphi)(0) = \mathbb{R}$, but

$$(\mathcal{P}_{\mathrm{Gr}\varphi})(0, \varphi(0)) = \big\{(\cos t, \sin t) : t \in [0, 2\pi]\big\} \subset \mathbb{R}^2.$$

In the literature (cf. [1, 5, 6]) the paratingent was considered only as a set-valued function acting from $\mathbb{R}$ into a family of non-empty subsets of $\mathbb{R}^n$. Instead in this note we characterize the set $(\mathcal{P}\varphi)(a)$ by the properties of φ.

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