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The natural operators of general affine connections into general affine connections

ABSTRACT. We reduce the problem of describing all $\mathcal{M}f_m$-natural operators transforming general affine connections on m-manifolds into general affine ones to the known description of all $GL(\mathbf{R}^m)$-invariant maps $\mathbf{R}^{m*} \otimes \mathbf{R}^m \to \otimes^k \mathbf{R}^{m*} \otimes \otimes^k \mathbf{R}^m$ for $k = 1, 3$.

Introduction. All manifolds considered in this paper are assumed to be finite dimensional, without boundaries, second countable, Hausdorff and smooth (of class C^∞). Maps between manifolds are assumed to be smooth (of class C^∞). The category of m-dimensional manifolds and their embeddings is denoted by $\mathcal{M}f_m$.

A classical linear connection on a manifold M is a right invariant connection Γ on the principal fiber bundle LM of linear frames of M. It can be considered equivalently as the corresponding $\mathbf{R}$-bilinear map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ such that $\nabla_{fX} Y = f\nabla_X Y$ and $\nabla_X fY = X(f)Y + f\nabla_X Y$ for any map $f : M \to \mathbf{R}$ and any vector fields $X, Y \in \mathcal{X}(M)$ on M, see [2].

A general affine connection on M is a right invariant connection Γ on the principal fiber bundle AM of affine frames of M. It can be equivalently considered as the corresponding pair (∇, K) consisting of a classical linear connection ∇ on M and a tensor field K of type $(1, 1)$ on M, see [2].

The general concept of natural operators can be found in [3].

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In the present note, we study the problem of finding all $\mathcal{M}f_m$-natural operators $B : Q_{gen\text{-}af} \rightsquigarrow Q_{gen\text{-}af}$ transforming general affine connections (∇, K) on m-manifolds M into general affine connections $B(\nabla, K)$ on M.

Given an $\mathcal{M}f_m$-natural operator $B : Q_{gen\text{-}af} \rightsquigarrow Q_{gen\text{-}af}$, we define an $\mathcal{M}f_m$-natural operator $\Delta : Q_{gen\text{-}af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$ by

$$B(\nabla, K) = (\nabla, K) + \Delta(\nabla, K)$$

for all general affine connections (∇, K) on m-manifolds M, and vice versa. So, to find all $\mathcal{M}f_m$-natural operators $B : Q_{gen\text{-}af} \rightsquigarrow Q_{gen\text{-}af}$ it is sufficient to find all $\mathcal{M}f_m$-natural operators $\Delta : Q_{gen\text{-}af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$ transforming general affine connections (∇, K) on m-manifolds M into pairs $\Delta(\nabla, K) = (\Delta^1(\nabla, K), \Delta^2(\nabla, K))$ of tensor fields $\Delta^1(\nabla, K)$ of type $(1, 2)$ and $\Delta^2(\nabla, K)$ of type $(1, 1)$ on M.

In the present note, we prove that the above problem of finding all $\mathcal{M}f_m$-natural operators $B : Q_{gen\text{-}af} \rightsquigarrow Q_{gen\text{-}af}$ (or $\Delta : Q_{gen\text{-}af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$) can be reduced to the one of describing all $GL(\mathbf{R}^m)$-invariant maps $\mathbf{R}^{m*} \otimes \mathbf{R}^m \to \otimes^k \mathbf{R}^{m*} \otimes \otimes^k \mathbf{R}^m$ for $k = 1, 3$.

This "reduction" is satisfactory, because the $GL(\mathbf{R}^m)$-invariant maps $\mathbf{R}^{m*} \otimes \mathbf{R}^m \to \otimes^k \mathbf{R}^{m*} \otimes \otimes^k \mathbf{R}^m$ for $k = 1, 2, 3$ are described in [1].

1. The crucial lemma. We prove the following lemma.

Lemma 1. *There is the bijection between the set C of all $\mathcal{M}f_m$-natural operators $\Delta : Q_{gen\text{-}af} \rightsquigarrow (\otimes^2 T^* \otimes T) \oplus (T^* \otimes T)$ and the set D of all $GL(\mathbf{R}^m)$-invariant maps $(\bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \to (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m)$.*

Proof. We define a map $\Phi : C \to D$ as follows.

Any $\Delta \in C$ is determined by the values

$$\Delta(\nabla, K)(x) = (\Delta^1(\nabla, K)(x), \Delta^2(\nabla, K)(x)) \in (\otimes^2 T^*_x M \otimes T_x M) \oplus (T^*_x M \otimes T_x M)$$

for all m-manifolds M, all linear connections ∇ on M, all tensor fields K of type $(1, 1)$ on M and all $x \in M$. Because of the $\mathcal{M}f_m$-invariance of Δ, we may assume that $M = \mathbf{R}^m$, $x = 0$. We can even assume that $id_{\mathbf{R}^m}$ is ∇-normal with center 0 (then $\nabla(0) \in \bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m$ because the Christoffel symbols ∇^i_{jk} of ∇ satisfy $\nabla^i_{jk}(0) + \nabla^i_{kj}(0) = 0$). Then using the invariance of Δ with respect to the homotheties $a_t = t\, id_{\mathbf{R}^m}$ for $t > 0$, we obtain the homogeneity condition

$$\Delta((a_t)_*\nabla, (a_t)_*K)(0) = (t\Delta^1(\nabla, K)(0), \Delta^2(\nabla, K)(0)) .$$

Because of the homogeneous function theorem [3], this type of the homogeneity implies that $\Delta(\nabla, K)(0)$ depends on $\nabla(0)$ and $j^1_0 K$ (only). Let $(\Lambda, \tau_0, \tau_1) \in (\bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \tilde{=} (\bigwedge^2 T^*_0 \mathbf{R}^m \otimes$

$T_0\mathbf{R}^m) \oplus J^1_0(T^*\mathbf{R}^m \otimes T\mathbf{R}^m)$, where $\tilde{=}$ is the usual $GL(\mathbf{R}^m)$-invariant identification. We put

$$\Phi(\Delta)(\Lambda, \tau_0, \tau_1) := \Delta(\nabla, K)(0) \in (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m)$$

(modulo the usual $GL(\mathbf{R}^m)$-invariant identification), where ∇ is the linear connection on $\mathbf{R}^m$ such that the Christoffel symbols of ∇ with respect to the chart $id_{\mathbf{R}^m}$ are constant maps and $\nabla(0) = \nabla^o(0)+\Lambda$ and ∇^o is the usual flat torsion free connection on $\mathbf{R}^m$ and K is the tensor field of type $(1,1)$ on $\mathbf{R}^m$ such that the coefficients of K in the chart $id_{\mathbf{R}^m}$ are polynomials of degree not more than 1 and $j^1_0K = (\tau_0, \tau_1)$.

Since Δ is determined by $\Phi(\Delta)$, Φ is injective.

It remains to show that Φ is surjective. Let $c : (\bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \to (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m)$ be a $GL(\mathbf{R}^m)$-invariant map (an element from D). Using the usual $GL(\mathbf{R}^m)$-invariant identification $\mathbf{R}^m = T_0\mathbf{R}^m$, we have the $GL(\mathbf{R}^m)$-invariant map

$$c : (\bigwedge^2 T^*_0\mathbf{R}^m \otimes T_0\mathbf{R}^m) \oplus (J^1_0(T^*\mathbf{R}^m \otimes T\mathbf{R}^m)) \to$$
$$\to (\otimes^2T^*_0\mathbf{R}^m \otimes T_0\mathbf{R}^m) \oplus (T^*_0\mathbf{R}^m \otimes T_0\mathbf{R}^m)\,.$$

Let (∇, K) be a general connection on an m-manifold M. Using c, we define a pair $\Delta_c(\nabla, K)$ consisting of tensor fields $\Delta^1_c(\nabla, K)$ of type $(1,2)$ and $\Delta^2_c(\nabla, K)$ of type $(1,1)$ on M as follows. Let $x \in M$. Consider a normal coordinate system φ of ∇ with center x. Then $(\varphi_*\nabla)_0 \in \bigwedge^2 T^*_0\mathbf{R}^m \otimes T_0\mathbf{R}^m$ modulo the obvious $GL(\mathbf{R}^m)$-invariant identification and $j^1_0(\varphi_*K) \in J^1_0(T^*\mathbf{R}^m \otimes T\mathbf{R}^m)$. We put

$$(\varphi_*\Delta_c(\nabla, K))_0 := c((\varphi_*\nabla)_0, j^1_0(\varphi_*K))\,.$$

If ψ is another normal coordinate system of ∇ with center x, then $\psi = \eta\circ\varphi$ for a $GL(\mathbf{R}^m)$-map η. Then $(\psi_*\Delta_c(\nabla, K))_0 = (\varphi_*\Delta_c(\Delta, K))_0$ because of the $GL(\mathbf{R}^m)$-invariance of c. That is why, the definition of $\Delta_c(\nabla, K)$ is correct. Thus we have the $\mathcal{M}f_m$-natural operator $\Delta_c : Q_{gen\text{-}af} \rightsquigarrow (\otimes^2T^* \otimes T) \oplus (T^* \otimes T)$. Clearly, $\Phi(\Delta_c) = c$. □

2. The main result. The main result of the note is the following "reduction" theorem.

Theorem 1. *The problem of finding all $\mathcal{M}f_m$-natural operators $B : Q_{gen\text{-}af} \rightsquigarrow Q_{gen\text{-}af}$ can be reduced to the one of describing all $GL(\mathbf{R}^m)$-invariant maps $\mathbf{R}^{m*} \otimes \mathbf{R}^m \to \otimes^k\mathbf{R}^{m*} \otimes \otimes^k\mathbf{R}^m$ for $k = 1, 3$.*

Proof. Any $GL(\mathbf{R}^m)$-invariant map $c : (\bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \to (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m)$ is the system of $GL(\mathbf{R}^m)$-invariant maps

$$c_1 : (\bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m) \to \otimes^2\mathbf{R}^{m*} \otimes \mathbf{R}^m$$

and

$$c_2 : (\bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\mathbf{R}^{m*} \otimes \mathbf{R}^m) \oplus (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \to \mathbf{R}^{m*} \otimes \mathbf{R}^m\,.$$

Using the invariance of c_i with respect to the homotheties $a_t = t\,id_{\mathbf{R}^m}$ for $t > 0$, we obtain the respective homogeneity conditions. Then (by the homogeneous function theorems) $c_1(\Lambda, \tau_0, \tau_1)$ is linear in Λ and τ_1 and not necessarily linear in τ_0. Then c_1 can be treated as the sum of $GL(\mathbf{R}^m)$-linear maps

$$c_1' : \mathbf{R}^{m*} \otimes \mathbf{R}^m \to (\bigwedge^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m)^* \otimes (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \subset \otimes^3 \mathbf{R}^{m*} \otimes \otimes^3 \mathbf{R}^m$$

and

$$c_1'' : \mathbf{R}^{m*} \otimes \mathbf{R}^m \to (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m)^* \otimes (\otimes^2 \mathbf{R}^{m*} \otimes \mathbf{R}^m) \tilde{=} \otimes^3 \mathbf{R}^{m*} \otimes \otimes^3 \mathbf{R}^m\,.$$

By the same arguments, $c_2(\Lambda, \tau_0, \tau_1)$ is independent of Λ and τ_1. Then $c_2 : \mathbf{R}^{m*} \otimes \mathbf{R}^m \to \mathbf{R}^{m*} \otimes \mathbf{R}^m$ is a $GL(\mathbf{R}^m)$-invariant map.

Now, Theorem 1 is an immediate consequence of Lemma 1. □

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