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RAYMOND MORTINI

The Riemann–Cantor uniqueness theorem for unilateral trigonometric series via a special version of the Lusin–Privalov theorem

Abstract. Using Baire's theorem, we give a very simple proof of a special version of the Lusin–Privalov theorem and deduce via Abel's theorem the Riemann–Cantor theorem on the uniqueness of the coefficients of pointwise convergent unilateral trigonometric series.

1. Introduction.

The earliest uniqueness theorem for trigonometric functions, postulated by Riemann and proved by Cantor reads as follows ([7, p. 326, Theorem 3.1, Chap. IX, Vol. I]):

Theorem 1.1 (Riemann–Cantor). *If the trigonometric series* $\sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ *converges for all* $\theta \in \mathbb{R}$ *to* 0*, then* $a_n = 0$ *for all* $n \in \mathbb{Z}$.

The proof is, in our viewpoint, rather tricky and technical. It is the aim of this note to give, for the unilateral trigonometric series, a simple, quite elementary proof which is mainly based on Baire's theorem. To achieve our goal, we present a simple proof of a very special case of the Lusin–Privalov theorem [5] on boundary values of functions holomorphic in the disk. For a nice survey on these uniqueness theorems, we refer to [3].

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2. A special case of the Lusin–Privalov uniqueness theorem and the Riemann–Cantor theorem. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ its boundary. The space of all bounded holomorphic functions in $\mathbb{D}$ is denoted, as usual, by $H^\infty := H^\infty(\mathbb{D})$. One of the earliest theorems in function theory of the disk, and which used Lebesgue's theory, stems from Fatou [2] and tells us that every $f \in H^\infty$ admits radial limits $f^*(e^{it}) := \lim_{r\to 1} f(re^{it})$ almost everywhere. A short time later, the Riesz brother's [4] showed that if $f^* = 0$ on a set of positive Lebesgue measure, then $f \equiv 0$. G. Szegö realized that actually $\log |f^*| \in L^1(\mathbb{T})$ if $f \not\equiv 0$, $f \in H^\infty$. This can be seen in the following way whenever $f(0) \neq 0$ and $||f||_\infty \le 1$: since $\log |f|$ is subharmonic,

$$\log |f(0)| \le \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| dt.$$

Now we apply Fatou's lemma to the functions $p_n(t) = -\log |f(r_n e^{it})|$, where $r_n \to 1$ is chosen so that f has no zero on the circles of radii r_n. Thus, Fatou's inequality $\int \liminf p_n \le \liminf \int p_n$, $p_n \ge 0$, yields

$$\log |f(0)| \le \frac{1}{2\pi} \int_0^{2\pi} \log |f^*(e^{it})| dt. \tag{2.1}$$

We are now ready to prove the "baby" version of the Lusin–Privalov theorem (see also [1, p. 12]):

Theorem 2.1. *Let f be holomorphic in $\mathbb{D}$ and suppose that* $\lim_{r\to 1} f(re^{i\theta}) = 0$ *for every $\theta \in \mathbb{R}$. Then $f \equiv 0$.*

Proof. Consider the set of continuous functions $u_\theta : [0,1] \to \mathbb{C}$ given by $u_\theta(r) := f(re^{i\theta})$. Each of these functions is bounded. So

$$\mathbb{T} = \bigcup_{n=1}^{\infty} \left\{e^{i\theta} : |u_\theta| \le n \text{ on } [0,1]\right\}$$

is a countable union of closed sets. By Baire's theorem, there is n_0 such that

$$\left\{e^{i\theta} : |u_\theta| \le n_0 \text{ on } [0,1]\right\}$$

contains an open arc $I \subseteq \mathbb{T}$. Let J be a closed arc with the same center as I with $J \subseteq I$. Then f is bounded on the sector $S = \{z \in \mathbb{D} : z/|z| \in J\}$ and f has radial limit 0 everywhere on I. Let $U := S^\circ$. Using a suitable rotation, we may assume that $U = \{z \in \mathbb{D} : 0 < \arg z < \alpha\}$. Map U by a conformal map ϕ onto the unit disk; we may take

$$\phi = \frac{z-i}{z+i} \circ z^2 \circ \frac{1+z}{1-z} \circ z^{\frac{\pi}{\alpha}}.$$

Note that ϕ has a holomorphic extension to J°. Let $\tilde{J} = \phi(J)$. Then any ray in U ending at a point in J goes to a simple curve γ tending to a point on $\tilde{J}$. So $g := f \circ \phi^{-1} \in H^\infty(\mathbb{D})$ and tends to 0 on γ. By Lindelöf's theorem (for a short, elegant and easy proof, see [6, p. 259]) g tends radially to 0 along every radius ending at $\tilde{J}^\circ$. Formula (2.1) now implies that $g = f \circ \phi^{-1}$ must be the zero function in $\mathbb{D}$ and so $f \equiv 0$ in $\mathbb{D}$ (note that we do not have to use Fatou's theorem on the existence of radial limits, since it is *assumed* that f admits these limits and that ϕ^{-1} definitely has an analytic extension at all, but three points of $\mathbb{T}$). □

Here is yet a more elementary approach, communicated to me by Robert Burckel, which does not even use formula (2.1) and the notion of subharmonicity.

We know from the proof above that the function $g \in H^\infty$ has radial limits 0 at *every point* of the open arc $\tilde{J}^\circ$. Taking suitable rotations of this arc, we arrive at a function $G(z) = \prod_{j=1}^m g(e^{i\theta_j} z)$, $G \in H^\infty$, that has radial limit 0 everywhere. Computing the Taylor coefficients b_n of G using the formula

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} G(se^{it}) s^{-n} e^{-int}\, dt, \quad 0 < s < 1,$$

we may use Lebesgue's dominated convergence theorem with $s \to 1$ to get $b_n = 0$ for every $n \in \mathbb{N}$. Hence G and therefore $f \equiv 0$.

Corollary 2.2 (Unicity theorem). *Let $S(\theta) \sim \sum_{n\in\mathbb{N}} a_n e^{in\theta}$ be a (one sided) trigonometric series. Suppose that for every $\theta \in \mathbb{R}$ the series converges to 0. Then $a_n = 0$ for every $n \in \mathbb{N}$.*

Proof. Associate with S the series $f(z) := \sum_{n\in\mathbb{N}} a_n z^n$. Since $S(0) = \sum_{n\geq 0} a_n$ converges, $a_n \to 0$. Thus the radius of convergence of f is at least 1. Hence $f \in H(\mathbb{D})$. Abel's theorem implies that

$$\lim_{r\to 1} f(re^{i\theta}) = S(\theta) = 0$$

for every θ. Hence, by Theorem 2.1, $f \equiv 0$ in $\mathbb{D}$ and so $a_n = 0$ for every n. □

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Raymond Mortini
Université de Lorraine
Département de Mathématiques et
Institut Élie Cartan de Lorraine, UMR 7502
Ile du Saulcy
F-57045 Metz, France
e-mail: `raymond.mortini@univ-lorraine.fr`

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