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INTRINSIC SYMMETRIES

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ABSTRACT

In this paper a concept of symmetry in the parameter space of the parameter dependent Hamiltonians is considered. The three different ways of realization of this symmetry is introduced. The example of analysis of this kind of symmetries is made in case of spherical harmonic oscillator. Some consequences of this symmetry for the electric type transition amplitudes of the electromagnetic nuclear radiation is shown.

1. INTRODUCTION

One of the most important topics of nuclear physics, and generally of the many-body problems, is an idea of intrinsic frame. The fixed-body frame can be defined as the frame which is attached to a nucleus. Description in an intrinsic frame is very useful because it leads to separation of the bulk degrees of freedom which are connected with external motion, like rotation or translations, from intrinsic motion, as for example nuclear vibrations. A rather general definition of fixed-body frame is described by Biedenharn and Louck [1]. Another idea of intrinsic frame is based on a separation of various types of quantum motions. It was successfully analyzed by Eckart in molecular physics [2]. However, it was proved by Guichardet [3] that both kinds of motion, i.e., rotation and vibration, cannot be, in general, separated exactly. From the technical point of view calculations in intrinsic frame are more effective, however, there are some complications coming from lack of uniqueness of transformations from the laboratory frame to the intrinsic frame [4, 5, 6, 7].

In this paper we consider implications of some specific symmetries which we consider in the intrinsic frame.

2. INTRINSIC SYMMETRY IN A SPACE OF PARAMETERS

A classical definition of symmetry operations base on an idea of invariance of a Hamiltonian, in a global sense. The operators which leave invariant the Hamiltonian $\hat{H}$ furnish the symmetry group which are understood as automorphisms $g \in \mathrm{Aut}(\mathcal{K})$ of space of quantum states $\mathcal{K}$. The invariance condition can be written as:

$$\hat{g}\hat{H}\hat{g}^{-1} = \hat{H}. \tag{1}$$

Existing of this kind of regularities in the structure of a Hamiltonian is a very valuable feature because it leads to some statements about properties of this Hamiltonian. But one can discern not only these classical symmetries but also some other regularities in the structure of the Hamiltonian.

For a class of Hamiltonians, or operators in general, which possess dependence on a set of parameters modelling a physical system, one can find symmetries which are related to the space of these parameters. A typical example of Hamiltonians of this kind can be found in the mean field theories. The standard problem considered in this theory is analysing of the mean field Hamiltonian dependent on some deformation parameters.

For illustration of this idea let us consider a set of parameters α belonging to a given domain $\alpha \in \mathbf{X}_{def}$. These parameters allow to model the physical system by a family of Hamiltonians $\hat{H}(\alpha)$. Let us consider the group G_α which is a subgroup of the full group of automorphisms of the parameter space $\mathbf{X}_{def}$, i.e. $\mathrm{G}_\alpha \subset \mathrm{Aut}(\mathbf{X}_{def})$. Due to the spectral theorem, the Hamiltonian (we assume only the discrete spectrum here) can be described as follows:

$$\hat{H}(\alpha) = \sum_{\nu} \epsilon_\nu(\alpha)\hat{P}_\nu(\alpha), \tag{2}$$

where the operator $\hat{P}_\nu(\alpha)$ is the projection operator

$$\hat{P}_\nu(\alpha) = \sum_{\mu} |\alpha; \nu\mu\rangle\langle\alpha; \nu\mu| \tag{3}$$

and the states $|\alpha; \nu\mu\rangle$ are parameter dependent eigenstates of the Hamiltonian. In this case, one can find a kind of symmetries, but not in respect to the variables of the model but in the parameter space $\mathbf{X}_{def}$.

Because the parameters α are not the variables (degrees of freedom) of this model, the Hamiltonian $\hat{H}(\alpha)$ should be understood as an operator valued function of the parameters α. For this reason the action of elements of the group G_α on the Hamiltonian is not that for operators but it coincides with this for functions:

$$\forall_{g_\alpha \in G_\alpha} \; \hat{g}_\alpha \hat{H}(\alpha) = \hat{H}(g_\alpha^{-1}\alpha). \tag{4}$$

This kind of transformations change both: the projection operators

$$\hat{g}_\alpha \hat{P}_\nu(\alpha) = \hat{P}_\nu(g_\alpha^{-1}\alpha) \tag{5}$$

and the eigenvalues

$$\hat{g}_\alpha \epsilon_\nu(\alpha) = \epsilon_\nu(g_\alpha^{-1}\alpha). \tag{6}$$

independently.

Naturally these symmetries do not lead to the same conclusions as the global symmetries related to the variables of the physical system. They rather show the structure of the Hamiltonian while changing the parameters α.

The symmetry in the parameter space can be realized in three different ways.

1. Case 1. The group $G_\alpha = \mathrm{Sym}_\alpha(\epsilon_\nu(\alpha))$ is a symmetry group of the eigenenergies as functions of the parameters α, but it is not the symmetry group for the projection operators $\hat{g}\hat{P}_\nu(\alpha)$:

$$\hat{g}\epsilon_\nu(\alpha) = \epsilon_\nu(g^{-1}\alpha) = \epsilon_\nu(\alpha), \qquad \hat{g}\hat{P}_\nu(\alpha) \neq \hat{P}_\nu(\alpha). \tag{7}$$

In this case, we can expect the same energy spectra for each configuration of the parameters α which differ by any transformation $g \in G_\alpha$, i.e. the energy spectrum of the physical system described in this way is the same for α and $\hat{g}\alpha$. It has to be noted that though these spectra are the same, the corresponding states/eigenvectors of both energy bands are different.

2. Case 2. The group $G_\alpha = \mathrm{Sym}_\alpha(\hat{P}_\nu(\alpha))$ is the symmetry group of the projection operators but it is not the symmetry group of the corresponding eigenenergies:

$$\hat{g}\epsilon_\nu(\alpha) \neq \epsilon_\nu(\alpha), \qquad \hat{g}\hat{P}_\nu(\alpha) = \hat{P}_\nu(g^{-1}\alpha) = \hat{P}_\nu(\alpha). \tag{8}$$

In this case, any transformation of the eigensolutions by an arbitrary element of the group $g \in G_\alpha$ gives the vector belonging to the same eigenspace. The action of the symmetry group G_α is closed within the eigenspace as it is observed for standard type of the Hamiltonian symmetries.

3. Case 3. In the third case, the group G_α is the common symmetry group for eigenspaces and eigenvalues. Such group can be understood as the global symmetry group acting in the parameter space of the full Hamiltonian, $\mathrm{Sym}_\alpha(\hat{H}(\alpha)) = G_\alpha$.

$$\hat{g}\hat{H}(\alpha) = \hat{H}(g^{-1}\alpha) = \hat{H}(\alpha) \tag{9}$$

$$\hat{g}\epsilon_\nu(\alpha) = \epsilon_\nu(g^{-1}\alpha) = \epsilon_\nu(\alpha) \tag{10}$$

$$\hat{g}\hat{P}_\nu(\alpha) = \hat{P}_\nu(g^{-1}\alpha) = \hat{P}_\nu(\alpha). \tag{11}$$

In this case the periodical behavior, with respect to the elements of the group G_α, can be seen in both: eigenenergies and projection operators.

Searching for these kinds of symmetries is not easy. In a natural way, it can be done only in the last case because in this case we do not need to decompose the Hamiltonian according to the spectral theorem. On the other hand, if the symmetries in the parameter space are already known, they allow to simplify calculations of expressions related to this Hamiltonian. Usually, it is sufficient to solve the required problem on the subset of $\mathbf{X}_{def}$ consisted of representative of the orbits constructed with respect to the action of the symmetry group G_α. The solutions for the other values of the parameters $\alpha \in \mathbf{X}_{def}$ can be obtained by acting with the elements of the group G_α on the required solutions got for the representatives of the constructed orbits.

The next section contains a simple example of analysis of symmetry on the parameters space.

3. EXAMPLE

The problem we are going to consider in this section is the parameters space symmetry in a case of a schematic model represented by the spherical harmonic oscillations. Let us assume the Hamiltonian of spherical harmonic oscillator, where the mass parameter $B \in \mathbb{R}_+$ and the stiffness parameter $C \in \mathbb{R}_+$ form the space of parameters $\mathbf{X}_{def}$.

$$\hat{H}(B, C) = -\frac{\hbar^2}{2B}\Delta + \frac{1}{2}C\,r^2. \tag{12}$$

The eigensolutions for this Hamiltonian are well known and can be written as [8]:

$$\Phi_{n\lambda\mu}(B, C;\, r, \theta, \phi) = N_{n\lambda}(B, C)\; r^\lambda \exp(-\nu r^2) L_n^{\lambda+1/2}(2\nu r^2) Y_{\lambda\mu}(\theta, \phi). \tag{13}$$

$$E_{\lambda\mu}(B, C) = \hbar\sqrt{\frac{C}{B}}\left(2n + \lambda + \frac{3}{2}\right). \tag{14}$$

The functions $L_n^\lambda(z)$ are the generalized Laguerre polynomials, the constants $N_{n\lambda}$ and ν are defined as:

$$N_{n\lambda}(B,C) = \left[\frac{2(2\nu)^{\lambda+\frac{3}{2}}n!}{\Gamma(\lambda+n+\frac{3}{2})}\right]^{\frac{1}{2}}, \quad \nu = \frac{\sqrt{CB}}{2h}. \tag{15}$$

The subgroup of automorphisms considered in this example is the group of scaling transformations. The action of an element of this group in the space of parameters (B,C) is:

$$\hat{g}_{\xi,\eta}(B,C) = (\xi B, \eta C). \tag{16}$$

The action of this group on the eigensolutions of the harmonic oscillator can be directly calculated and expressed in the following form:

$$\begin{aligned} \hat{g}_{\xi\eta}E_{n\lambda}(B_0,C_0) &= E_{n\lambda}(\xi^{-1}B_0,\eta^{-1}C_0) = \\ &\left(\frac{\xi}{\eta}\right)^{\frac{1}{2}} E_{n\lambda}(B_0,C_0). \\ \hat{g}_{\xi\eta}\Psi(B_0,C_0;\,r,\theta,\phi) &= \Psi(\xi^{-1}B_0,\eta^{-1}C_0;\,r,\theta,\phi) = \\ &(\xi\eta)^{-\frac{3}{8}}\,\Psi\left(B_0,C_0;(\xi\eta)^{-\frac{1}{4}}r,\theta,\phi\right). \end{aligned} \tag{17}$$

The first and the second case described in the previous section can be now written as:

1. The first case one can find by taking $\xi = \eta = \chi$:

$$\begin{aligned} &\hat{g}_{\chi\chi}E_{n\lambda}(B_0,C_0) = E_{n\lambda}(B_0,C_0), \\ &\hat{g}_{\chi\chi}\Psi(B_0,C_0;\,r,\theta,\phi) = (\sqrt{\chi})^{-\frac{3}{4}}\,\Psi\left(B_0,C_0;(\chi)^{-\frac{1}{2}}r,\theta,\phi\right). \end{aligned} \tag{18}$$

2. The second case is obtained by assuming $\xi = 1/\eta = \varepsilon$

$$\begin{aligned} &\hat{g}_{\varepsilon\ 1/\varepsilon}E_{n\lambda}(B_0,C_0) = \varepsilon\ E_{n\lambda}(B_0,C_0), \\ &\hat{g}_{\varepsilon\ 1/\varepsilon}\Psi(B_0,C_0;\,r,\theta,\phi) = \Psi\left(B_0,C_0;r,\theta,\phi\right). \end{aligned} \tag{19}$$

3. The third case corresponds to the trivial symmetry group consisted of the identity operation $\xi = \eta = 1$.

An interesting question is to analyse implications of an action of the scaling group in the parameter space onto the electromagnetic transition amplitudes. According to Bohr and Mottelson [9] the transition amplitude of a photon, of a given multipole type, is proportional to the difference of energies between the final and the initial state raised to the appropriate power and multiplied by the reduced transition probability. More explicitly the formula is given by:

$$T(E(M)\lambda;\,I_1 \to I_2) = \mathcal{B}(\lambda)(E_2 - E_1)^{2\lambda+1}B(E(M)\lambda;\,I_1 \to I_2), \tag{20}$$

where the reduced transition probability is obtained by means of the Wigner-Eckart theorem:

$$B(E(M)\lambda;\, I_1 \to I_2) = \sum_{\mu M_2} |\langle I_2 M_2|\mathcal{M}(E(M)\lambda, \mu)|I_1 M_1\rangle|^2 . \tag{21}$$

The single-particle Hamiltonian (12) can describe a system of independent nucleons moving in the potential of harmonic oscillator. Let us consider only the transition of the electric type. The schematic multipole transition operator can be written in the simple form:

$$\mathcal{M}(E\lambda, \mu) = \mathcal{A}\, r^{\lambda} Y_{\lambda\mu}(\theta, \phi). \tag{22}$$

Using the previous formulae, we can obtain the following scaling dependence of the matrix elements of the multipole transition operators:

$$\begin{aligned}&\langle n_2, I_2, M_2;\, \xi B_0, \eta C_0|\mathcal{M}(E\lambda, \mu)|n_1, I_1, M_1;\, \xi B_0, \eta C_0\rangle = \\ &(\xi\eta)^{-\frac{\lambda}{4}}\, \langle n_2, I_2, M_2;\, B_0, C_0|\mathcal{M}(E\lambda, \mu)|n_1, I_1, M_1;\, B_0, C_0\rangle.\end{aligned} \tag{23}$$

It implies that the scaled reduced probabilities and the amplitudes can be expressed as:

$$B(E\lambda;\, \xi B_0, \eta C_0; I_1 \to I_2) = (\xi\eta)^{-\frac{\lambda}{2}} B(E\lambda;\, B_0, C_0; I_1 \to I_2), \tag{24}$$

$$\begin{aligned}&T(E\lambda; \xi B_0, \eta C_0; I_1 \to I_2) = \\ &\left(\frac{\eta}{\xi}\right)^{\lambda+\frac{1}{2}} (\xi\eta)^{-\frac{\lambda}{2}} T(E\lambda; B_0, C_0; I_1 \to I_2).\end{aligned} \tag{25}$$

The special cases of symmetries in the parametric space considered earlier we obtain by the appropriate substitution of the scaling group parameters:

1. The first case is obtained for $\xi = \eta = \chi$

$$T(E\lambda; \chi B_0, \chi C_0; I_1 \to I_2) = \chi^{\lambda}\, T(E\lambda; B_0, C_0; I_1 \to I_2). \tag{26}$$

2. The second case is obtained for $\xi = 1/\eta = \varepsilon$

$$T(E\lambda; \varepsilon B_0, 1/\varepsilon C_0; I_1 \to I_2) = \varepsilon^{2\lambda+1}\, T(E\lambda; B_0, C_0; I_1 \to I_2). \tag{27}$$

The previous outcomes allow to conclude that changing the parameters of the scaling group which acts in the two dimensional space built from the pairs of numbers representing the mass and the stiffness parameters, leads to the scaling of the electric transitions amplitudes. The two cases of symmetry, described earlier, give the coefficients which are some powers of the scaling group parameters and which become the scaling factors of these amplitudes.

The results of the above analysis lead to a possibility of obtaining of all harmonic oscillators from the single Hamiltonian (12) taken as a pattern. This pattern can be obtained by fixing the parameters $B = B_0$ and $C = C_0$. After this procedure every spherical harmonic oscillator Hamiltonian can be described as an appropriate scaling of this standard/pattern oscillator. Obviously, this example is trivial, but using it we want to present, in a clear way, the idea of symmetry in the parametric space.

On the other hand, using the appropriate experimental data one can try to check if a given nuclear system has this kind of symmetry. Assume that two given nuclear energy bands fulfil the scaling properties as we found for the harmonic oscillator. In general, we need two conditions to measure the scaling parameters. It is sufficient to find the ratio of measured eigenenergy $E_{n\lambda}(B, C)$ of the second band and the corresponding energy $E_{n\lambda}(B_0, C_0)$ from the pattern oscillator spectrum using the formula (17):

$$\frac{E_{n\lambda}(B, C)}{E_{n\lambda}(B_0, C_0)} = \sqrt{\frac{\eta}{\xi}} = \frac{E_{n\lambda}(\xi B_0, \eta C_0)}{E_{n\lambda}(B_0, C_0)}. \tag{28}$$

This condition is sufficient to get the scaling parameter (only one is needed) in both special cases of symmetry (Case 1., Case 2.) considered earlier. However, in general, this ratio is not sufficient to get both parameters of the scaling group. Note that for our schematic model this ratio does not depend on quantum numbers n and λ.

To find the second condition, the next step is to calculate/measure a similar ratio for the transition amplitudes (25):

$$\frac{T(E\lambda; B, C; I_1 \to I_2)}{T(E\lambda; B_0, C_0; I_1 \to I_2)} = \left(\frac{\eta}{\xi}\right)^{\lambda+\frac{1}{2}} (\xi\eta)^{-\frac{\lambda}{2}} = \frac{T(E\lambda; \xi B_0, \eta C_0; I_1 \to I_2)}{T(E\lambda; B_0, C_0; I_1 \to I_2)}. \tag{29}$$

Using equations (28) and (29) one can get the group parameters ξ and η. The values of the constants ξ and η allow to describe the new oscillator as an appropriate scaling of the pattern oscillator. Having both parameters, one can describe properties of the second band if we know the structure of the first band.

In the special cases Case 1. defined by $\xi = \eta$ or Case 2. determined by $\xi = 1/\eta$, the new oscillator is related to the pattern oscillator as it is described in the cases Case 1. and Case 2. It is worth to note that all properties of any harmonic oscillator can be described by scaling of the appropriate expressions/properties found for the standard/pattern oscillator.

The above idea gives a possibility of existence of the so-called similar bands. In this context, the two bands are similar if properties of the other band can be obtained by applying the symmetry group (acting in the parameter space) on the properties of the first band. This leads to the characteristic relations among energy spectra of both bands and also some relations among transition probabilities within/between both bands.

Naturally the toy-model of spherical harmonic oscillator introduces very simple relations between the standard/pattern oscillator and any arbitrary harmonic oscillator Hamiltonian. Application of this idea to more realistic cases implies much more complex relations between the pattern and the corresponding model obtained for other parameters

The idea of symmetry in the space of parameters of a physical model belongs to the partial symmetries approach. Theses symmetries should be understood as a kind of relations connecting some specific features of a physical system driven by the parameters of this physical model. The tools required for analysis of symmetry in the parameters space are not well developed. But, as we show in this paper, this type of symmetry is an interesting idea which can be applied for qualitative, sometimes quantitative, analysis for some quantum systems.

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